Ph106c Book Notes

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Notation: G - Griffiths, Introduction to Electrodynamics, 4th Edition. J - Jackson, Classical Electrodynamics, 3rd edition. LN - Lecture Notes. HM - Heald and Marion, Classical Electromagnetic Radiation.

Maxwell's Equations (J 6.6, Pg 238)

$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

Effect of magnetic monopoles Maxwell's equations (where the unit of magnetic charge is the Ampere-meter):

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}$$
$$\nabla \cdot \boldsymbol{B} = \mu_0 \rho_m$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - \mu_0 \boldsymbol{j}_m$$
$$\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j}_e$$

and Lorentz force law:

$$\boldsymbol{F} = q_e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + q_m \left(\boldsymbol{B} - \boldsymbol{v} \times \frac{\boldsymbol{E}}{c^2} \right)$$

Misc vector identities LN Pg 596:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \nabla) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \nabla$$

Volume integrals (J Theorems from Vector Calculus) Let $\boldsymbol{n}$ be the outwardly directed normal unit vector.

$$\int_V \nabla \psi d^3x = \int_S \psi \boldsymbol{n} da$$
$$\int_V d^3x (\nabla \times \boldsymbol{A}) = \oint_S da (\hat{n} \times \boldsymbol{A})$$

Surface integrals (J Theorems from Vector Calculus)

$$\int_S \boldsymbol{n} \times \nabla \psi da = \oint_C \psi d\boldsymbol{l}$$

Charge conservation (J 5.2, Pg 175)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Under **magnetostatics**, we take $\frac{\partial \rho}{\partial t} = 0$, hence: $\nabla \cdot \boldsymbol{J} = 0$.

Biot-Savart Law (J 5.4, Pg 175)

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{l} \times \boldsymbol{x}}{|\boldsymbol{x}|^3}$$

where $\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{r}'$. Integrating (J 5.14, Pg 178):

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \boldsymbol{J}(\boldsymbol{x}') \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d^3x'$$

and by using the gradient of a reciprocal and integrating by parts (J 5.16, Pg 179),

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$

which is clearly divergence-free.

Infinite straight wire magnetic induction (J Pg 217)

$$B_\phi = \frac{\mu_0 I}{2\pi a} \frac{\rho_<}{\rho_>}$$

where $\rho_< = \min(a, \rho)$.

Force between two wires (J 5.10, Pg 178) The force on closed loop 1 due to closed loop 2 is:

$$\boldsymbol{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{(d\boldsymbol{l}_1 \cdot d\boldsymbol{l}_2)\boldsymbol{x}_{12}}{|\boldsymbol{x}_{12}|^3}$$

where $\boldsymbol{x}_{12}$ is the vector distance from $d\boldsymbol{l}_2$ to $d\boldsymbol{l}_1$.

Force and torque on current distribution (J 5.12-13, Pg 178)

$$\boldsymbol{F} = \int \boldsymbol{J}(\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x}) d^3x$$
$$\boldsymbol{N} = \int \boldsymbol{x} \times (\boldsymbol{J} \times \boldsymbol{B}) d^3x$$

Ampere's Law (Magnetostatics) (J 5.22, 5.25)

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$
$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$

Magnetic vector potential (J 5.27-28)

$$\boldsymbol{B}(\boldsymbol{x}) = \nabla \times \boldsymbol{A}(\boldsymbol{x})$$
$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' + \nabla \Psi(\boldsymbol{x})$$

where the gradient indicates the gauge freedom of the vector potential.

Gauge transformation (J 6.12-13, Pg 240)

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \Lambda$$
$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t}$$

Coulomb gauge (J Pg 181)

$$\nabla \cdot \boldsymbol{A} = 0$$

This gauge makes the scalar potential satisfy (J 6.22, Pg 241):

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

so that the scalar potential is the instantaneous Coulomb potential due to the charge density. The vector potential satisfies (J 6.30, Pg 242):

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}_t$$

where $\boldsymbol{J}_t$ is the transverse current. Note: V must be time independent in order to use the above equation (see LN Pg 638). The magnetic vector potential under the magnetostatic case becomes:

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$

Under the Coulomb gauge, the vector potential is continuous across boundaries (LN 5.74, Pg 304):

$$\boldsymbol{A}_2(\boldsymbol{r}) = \boldsymbol{A}_1(\boldsymbol{r})$$

The normal derivative of the normal component of $\boldsymbol{A}$ also does not change (LN 5.79, Pg 306):

$$\left.\frac{\partial (\boldsymbol{A}_1)_n}{\partial n}\right|_{\boldsymbol{r}} = \left.\frac{\partial (\boldsymbol{A}_2)_n}{\partial n}\right|_{\boldsymbol{r}}$$

The tangential derivatives of $\boldsymbol{A}$ are also continuous across a boundary (LN 5.84, Pg 308):

$$\hat{s} \cdot \nabla[\boldsymbol{A}_2(\boldsymbol{r}) - \boldsymbol{A}_1(\boldsymbol{r})] = 0$$

Lorentz Gauge (J 6.14, Pg 240) Choose $\nabla \cdot \boldsymbol{A}$ so that:

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$$

which gives the decoupled wave equations for the potentials (J 6.15-16, Pg 240):

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$
$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$$

We use the d'Alembertian with +,-,-,- metric:

$$\Box^2 V = \frac{\rho}{\epsilon_0}$$
$$\Box^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$
$$\Box^2 = \epsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \nabla^2$$

Current distributions Loop of radius a lying in xy plane (J 5.33, Pg 181):

$$J_\phi = I\sin\theta'\delta(\cos\theta')\frac{\delta(r'-a)}{a}$$

Current distribution of moving particles (J pg 187):

$$\boldsymbol{J} = \sum_i q_i \boldsymbol{v}_i \delta(\boldsymbol{x}-\boldsymbol{x}_i)$$
$$\boldsymbol{m} = \frac{1}{2}\sum_i q_i(\boldsymbol{x}_i \times \boldsymbol{v}_i) = \sum_i \frac{q_i}{2M_i}\boldsymbol{L}_i$$

Magnetic dipole field (J 5.41, Pg 183) far away from the loop

$$B_r = \frac{2\mu_0 m\cos\theta}{4\pi r^3}$$
$$B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

where $m = I\pi a^2$ is the magnetic dipole moment. In coordinate-free notation (J 5.56, Pg 186):

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\boldsymbol{m}) - \boldsymbol{m}}{|\boldsymbol{x}|^3}\right]$$

To account for the dipole itself, we can add a delta function contribution (J 5.64, Pg 188):

$$\frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\boldsymbol{m}) - \boldsymbol{m}}{|\boldsymbol{x}|^3} + \frac{8\pi}{3}\boldsymbol{m}\delta(\boldsymbol{x})\right]$$

If the current loop is not planar, the magnetic dipole moment will be smaller than if it were planar (LN 5.108, Pg 318):

$$\boldsymbol{m}_{loop} = \frac{I}{2}\oint_C (\boldsymbol{r}' \times d\boldsymbol{l}'(\boldsymbol{r}'))$$

Vector potential due to dipole (J 5.55, Pg 186)

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{|\boldsymbol{x}|^3}$$

Forces and torques on magnetic dipole (J 5.69, Pg 189)

$$\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$$

Note that this implies that the dipole likes to go to regions of high field. The torque is (J 5.71, Pg 190):

$$\boldsymbol{N} = \boldsymbol{m}\times\boldsymbol{B}(0)$$

giving a potential energy (J 5.72, Pg 190):

$$U = -\boldsymbol{m}\cdot\boldsymbol{B}$$

Legendre Polynomial derivative (J 5.47, Pg 184)

$$\frac{d}{dx}\left[\sqrt{1-x^2}P_l^1(x)\right] = l(l+1)P_l(x)$$

Magnetic moment density/Magnetization (J 5.53, Pg 186)

$$\boldsymbol{\mathcal{M}}(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}\times\boldsymbol{J}(\boldsymbol{x}))$$

so that the magnetic moment of a localized current distribution is the integral (J 5.54, Pg 186):

$$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{x}'\times\boldsymbol{J}(\boldsymbol{x}')d^3x'$$

Magnetization (J 5.76, Pg 192)

$$\boldsymbol{M}(\boldsymbol{x}) = \sum_i N_i\langle\boldsymbol{m}_i\rangle$$

Effective current density (J 5.79, Pg 192)

$$\boldsymbol{J}_M = \nabla\times\boldsymbol{M}$$

so that the curl of the macroscopic magnetic field is due to the flow of free charge and the effective current density due to the material magnetic moment (J 5.80-51, Pg 192):

$$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{J} + \nabla\times\boldsymbol{M})$$

Magnetic field (Macroscopic) (J 5.81, Pg 192)

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

so that the macroscopic equations are (J 5.82, Pg 193):

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}$$
$$\nabla\cdot\boldsymbol{B} = 0$$

where $\boldsymbol{J}$ refers to the free charge current density.

Linear magnetostatics (J 5.84, Pg 193)

$$\boldsymbol{B} = \mu\boldsymbol{H}$$
$$\boldsymbol{M} = \chi_m\boldsymbol{H}$$

where $\mu = \mu_r\mu_0$.

H and B field boundary conditions - general (J 5.86-87, Pg 194)

$$(\boldsymbol{B}_2 - \boldsymbol{B}_1)\cdot\boldsymbol{n} = 0$$
$$\boldsymbol{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{K}$$

where $\boldsymbol{n}$ points from region 1 to region 2. The second condition can be written (LN 5.72, Slide 302):

$$\boldsymbol{H}_2 - \boldsymbol{H}_1 = \boldsymbol{K}\times\boldsymbol{n}$$

H and B field boundary conditions - linear media (J 5.88-89, Pg 194)

$$\boldsymbol{B}_2\cdot\boldsymbol{n} = \boldsymbol{B}_1\cdot\boldsymbol{n}$$
$$\boldsymbol{B}_2\times\boldsymbol{n} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_1\times\boldsymbol{n}$$
$$\boldsymbol{H}_2\cdot\boldsymbol{n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_1\cdot\boldsymbol{n}$$
$$\boldsymbol{H}_2\times\boldsymbol{n} = \boldsymbol{H}_1\times\boldsymbol{n}$$

Note that for systems with spherical symmetry, these boundary conditions can be stated as (J 5.119, Pg 202):

$$H_\theta \text{ is continuous}$$
$$\implies \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{r_-} = \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{r_+}$$
$$B_r \text{ is continuous}$$
$$\implies \mu_-\left.\frac{\partial\Phi_M}{\partial r}\right|_{r_-} = \mu_+\left.\frac{\partial\Phi_M}{\partial r}\right|_{r_+}$$

Magnetic scalar potential (J 5.93, Pg 195) For $\boldsymbol{J} = 0$ in some finite region of space **that is simply connected**, define the scalar potential:

$$\boldsymbol{H} = -\nabla\Phi_M$$

For linear media, the divergence is given by $\nabla\cdot\boldsymbol{B} = 0 \implies \nabla\cdot\boldsymbol{H} = 0$ so Φ_M satisfies the Laplace Equation: $\nabla^2\Phi_M = 0$.

Multipole expansion of vector potential (Coulomb Gauge) (LN 5.91, Pg 312)

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{1}{r}\sum_{i=1}^{\infty}\frac{1}{r^l}\int_V d^3x'\boldsymbol{J}(\boldsymbol{r}')(r')^l P_l(\cos\gamma)$$

Note there is no $l = 0$ monopole term since there are no magnetic monopoles.

Hard Ferromagnets (J 5.95-98, Pg 196) $\boldsymbol{M}$ given, $\boldsymbol{J} = 0$. Then we have the magnetostatic Poisson equation:

$$\nabla^2\Phi_M = -\rho_M$$

and effective magnetic charge density:

$$\rho_M = -\nabla\cdot\boldsymbol{M}$$

and effective magnetic surface-charge density:

$$\sigma_M = \boldsymbol{n} \cdot \boldsymbol{M}$$

giving a scalar potential solution (J 5.100, Pg 197) with outward pointing normal:

$$\Phi_M(\boldsymbol{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla' \cdot \boldsymbol{M}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' + \frac{1}{4\pi}\oint_S \frac{\boldsymbol{n}'\boldsymbol{M}(\boldsymbol{x}')da'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

with asymptotic behavior of a dipole with $\boldsymbol{m} = \int \boldsymbol{M} d^3x$:

$$\Phi_M(\boldsymbol{x}) \to \frac{\boldsymbol{m}\cdot\boldsymbol{x}}{4\pi r^3}, \quad |\boldsymbol{x}-\boldsymbol{x}'| \to \infty$$

Vector potential of hard ferromagnets (J 5.103, Pg 197)

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' + \frac{\mu_0}{4\pi}\oint_S \frac{\boldsymbol{M}(\boldsymbol{x}')\times\boldsymbol{n}'}{|\boldsymbol{x}-\boldsymbol{x}'|}da'$$

Observe that $\nabla\times\boldsymbol{M}$ is the effective magnetic current density and $\boldsymbol{M}\times\boldsymbol{n}$ is the effective magnetic surface current density.

Uniformly magnetized sphere using magnetic scalar potential (J Pg 198) can be solved using the magnetic scalar potential from the surface magnetic-charge density:

$$\sigma_M = \boldsymbol{n}\cdot\boldsymbol{M} = M_0\cos\theta$$

giving a potential everywhere of (J 5.104, Pg 198):

$$\Phi_M(r,\theta) = \frac{1}{3}M_0 a^2 \frac{r_<}{r_>^2}\cos\theta$$

where $r_<, r_>$ are the minimum or maximum of (r, a) respectively. Inside the sphere:

$$\boldsymbol{H}_{in} = -\frac{1}{3}\boldsymbol{M}, \quad \boldsymbol{B}_{in} = \frac{2\mu_0}{3}\boldsymbol{M}$$

Outside the sphere, the potential is that of a dipole with:

$$\boldsymbol{m} = \frac{4\pi a^3}{3}\boldsymbol{M}$$

$$\boldsymbol{B}_{out} = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{r})\hat{r} - \boldsymbol{m}}{r^3}$$

Uniformly magnetized sphere using magnetic vector potential (J Pg 199) The volume current density $\nabla\times\boldsymbol{M}$ vanishes but the surface current density is non-vanishing:

$$\boldsymbol{M}\times\boldsymbol{n}' = M_0\sin\theta'\hat{\phi}$$

Letting the field point be in the xz plane $\phi = 0$, the vector potential only has an azimuthal component (J 5.109, Pg 199):

$$A_\phi(\boldsymbol{x}) = \frac{\mu_0}{4\pi}M_0 a^2 \int d\Omega' \frac{\sin\theta'\cos\phi'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

and expanding the distance term in spherical harmonics gives only the $l = 1, m = 1$ term that survives (J 5.111, Pg 200):

$$A_\phi(\boldsymbol{x}) = \frac{\mu_0}{3}M_0 a^2\left(\frac{r_<}{r_>^2}\right)\sin\theta$$

Magnetization of linear paramagnetic or diamagnetic substance (J 5.115, Pg 200) Inside an object placed in a uniform field $\boldsymbol{B}_0 = \mu_0\boldsymbol{H}_0$ (J 5.112, Pg 200),

$$\boldsymbol{B}_{in} = \boldsymbol{B}_0 + \frac{2\mu_0}{3}\boldsymbol{M}$$

$$\boldsymbol{H}_{in} = \frac{\boldsymbol{B}_0}{\mu_0} - \frac{\boldsymbol{M}}{3}$$

which for linear media $\boldsymbol{B}_{in} = \mu\boldsymbol{H}_{in}$ yields a magnetization (J 5.115, Pg 200):

$$\boldsymbol{M} = \frac{3}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\boldsymbol{B}_0$$

In general, and even for ferromagnets, the following relation holds (J 5.116, Pg 200):

$$\boldsymbol{B}_{in} + 2\mu_0\boldsymbol{H}_{in} = 3\boldsymbol{B}_0$$

Spherical shell magnetic shielding (J Pg 201-203) Let there be a spherical shell of inner and outer radii a, b of permeability μ placed in a uniform magnetic induction $\boldsymbol{B}_0$. The effect of the shell is to introduce a dipole field outside the shell with moment (J 5.121, Pg 202):

$$\alpha_1 = \left[\frac{(2\mu_r+1)(\mu_r-1)}{(2\mu_r+1)(\mu_r+2) - 2(\mu_r-1)^2a^3/b^3}\right] \cdot (b^3 - a^3)H_0$$

and the H-field inside the cavity is uniform, pointing in the direction of $\boldsymbol{H}_0$ with magnitude:

$$-\delta_1 = \frac{9\mu_r}{(2\mu_r+1)(\mu_r+2) - 2(\mu_r-1)^2a^3/b^3}H_0$$

In the limit $\mu \gg \mu_0$, we have:

$$\alpha_1 \to b^3H_0$$

$$-\delta_1 \to \frac{9\mu_0}{2\mu(1-a^3/b^3)}H_0$$

and the internal field vanishes as $1/\mu$ decreases.

Magnetic energy Let the vector potential change by $\delta\boldsymbol{A}(\boldsymbol{x})$ due to external sources. The work done by the external sources is (J 5.144, Pg 213):

$$\delta W = \int \delta\boldsymbol{A}\cdot\boldsymbol{J}d^3x$$

In terms of the magnetic field for localized field distributions (J 5.147, Pg 214):

$$\delta W = \int \boldsymbol{H}\cdot\delta\boldsymbol{B}d^3x$$

The total magnetic energy for linear media is (J 5.148, Pg 214):

$$W = \frac{1}{2}\int \boldsymbol{H}\cdot\boldsymbol{B}d^3x$$

which can be written as (J 5.149, Pg 214):

$$W = \frac{1}{2}\int \boldsymbol{J}\cdot\boldsymbol{A}d^3x$$

Energy change due to object in field Let the original medium have $\mu_0, \boldsymbol{B}_0$ and with the object in place, let the fields be $\boldsymbol{B}, \boldsymbol{H}$. Then the change in energy is:

$$W = \frac{1}{2}\int_{V_1}(\boldsymbol{B}\cdot\boldsymbol{H}_0 - \boldsymbol{H}\cdot\boldsymbol{B}_0)d^3x$$

and if μ_0 is the free space value, (J 5.150, Pg 214):

$$W = \frac{1}{2}\int_{V_1}\boldsymbol{M}\cdot\boldsymbol{B}_0 d^3x$$

Self and mutual inductance energy contribution (J 5.152, Pg 215)

$$W = \frac{1}{2}\sum_{i=1}^N L_i I_i^2 + \sum_{i=1}^N\sum_{j>i}^N M_{ij}I_iI_j$$

Explicitly, the coefficients are (J 5.154-155, Pg 215):

$$L_i = \frac{\mu_0}{4\pi I_i^2}\int_{C_i} d^3x_i \int_{C_i} d^3x_i' \frac{\boldsymbol{J}(\boldsymbol{x}_i)\cdot\boldsymbol{J}(\boldsymbol{x}_i')}{|\boldsymbol{x}_i - \boldsymbol{x}_i'|}$$

$$M_{ij} = \frac{\mu_0}{4\pi I_iI_j}\int_{C_i} d^3x_i \int_{C_j} d^3x_j' \frac{\boldsymbol{J}(\boldsymbol{x}_i)\cdot\boldsymbol{J}(\boldsymbol{x}_j')}{|\boldsymbol{x}_i - \boldsymbol{x}_j'|}$$

For planar circuits, we make the substitution $\boldsymbol{J}d^3x = \boldsymbol{J}_\| da d\boldsymbol{l}$ and write the expressions in terms of the vector potential. The end result is (J 5.156, Pg 216):

$$M_{ij} = \frac{1}{I_j}F_{ij}$$

where F_{ij} is the magnetic flux from circuit j linked within circuit i:

$$F_{ij} = \int_{S_i} \boldsymbol{B}_j \cdot \boldsymbol{n} da$$

Estimating self-inductance (J 5.157, Pg 216)

$$L = \frac{1}{I^2} \int \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{\mu} d^3x$$

Vector potential diffusion equation (J 5.160, Pg 219)

$$\nabla^2 \boldsymbol{A} = \mu\sigma \frac{\partial \boldsymbol{A}}{\partial t}$$

with characteristic time for the decay of an initial field configuration over a spatial lengthscale L (J 5.161, Pg 219):

$$\tau = O(\mu\sigma L^2)$$

Transverse and longitudinal current (J 6.27-28, Pg 242)

$$\boldsymbol{J}_l = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \boldsymbol{J}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$
$$\boldsymbol{J}_t = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{\boldsymbol{J}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$

so that:

$$\nabla \times \boldsymbol{J}_l = 0$$
$$\nabla \cdot \boldsymbol{J}_t = 0$$

Helmholtz wave equation (J 7.3, Pg 296) For EM waves with harmonic time dependence $e^{-i\omega t}$.

$$(\nabla^2 + \mu\epsilon\omega^2)\boldsymbol{E} = 0$$
$$(\nabla^2 + \mu\epsilon\omega^2)\boldsymbol{B} = 0$$

where $\mu = \mu_0\mu_r, \epsilon = \epsilon_0\epsilon_r$, giving dispersion relation (J 7.4, Pg 296):

$$k = \sqrt{\mu\epsilon}\omega$$

EM wave amplitude relations (J Pg 297) Let:

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{E}} e^{ik\boldsymbol{n}\cdot\boldsymbol{x} - i\omega t}$$
$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{\mathcal{B}} e^{ik\boldsymbol{n}\cdot\boldsymbol{x} - i\omega t}$$

Then (J 7.11, Pg 297):

$$\boldsymbol{\mathcal{B}} = \sqrt{\mu\epsilon}\boldsymbol{n} \times \boldsymbol{\mathcal{E}}$$
$$\boldsymbol{\mathcal{H}} = \boldsymbol{n} \times \boldsymbol{\mathcal{E}}/Z$$

where $Z = \sqrt{\frac{\mu}{\epsilon}}$ is the **impedance**. The free space impedance is 376.7Ω.
Further relations are (LN 9.23, Pg 494):

$$\boldsymbol{\mathcal{B}} = \frac{1}{v} \hat{k} \times \boldsymbol{\mathcal{E}}$$
$$\boldsymbol{\mathcal{E}} = -v\hat{k} \times \boldsymbol{\mathcal{B}}$$

Poynting vector (J Pg 298) The rate of mechanical work done by fields is (J 6.110, Pg 260):

$$\frac{dE_{mech}}{dt} = \int_V \boldsymbol{J} \cdot \boldsymbol{E} d^3x$$

and the total field energy is (6.112, Pg 260):

$$E_{field} = \frac{\epsilon_0}{2} \int_V (\boldsymbol{E}^2 + c^2\boldsymbol{B}^2) d^3x$$

The Poynting vector is defined (J 6.109, Pg 259):

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

and satisfies the energy conservation equation (J 6.108, Pg 259):

$$\frac{\partial u_{\text{field}}}{\partial t} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{E}$$

or in integral form (J 6.111, Pg 260):

$$\frac{d}{dt}(E_{mech} + E_{field}) = -\oint_S \boldsymbol{n} \cdot \boldsymbol{S} da$$

Its average value is:

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} \Re (\boldsymbol{E} \times \boldsymbol{H}^*)$$

For a plane wave,

$$\boldsymbol{S} = cu_{field}\hat{k}$$

Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$
$$\boldsymbol{K} = \sigma_{\square} \boldsymbol{E}$$

so that the rate of energy dissipation (rate of change of mechanical energy without fields coming in or out):

$$P_{mech} = \int_V \boldsymbol{J} \cdot \boldsymbol{E} d\tau$$
$$P_{mech} = \int_S \boldsymbol{K} \cdot \boldsymbol{E} da$$

Time-averaged energy density (J Pg 298)

$$u = \frac{1}{4} \left(\epsilon \boldsymbol{E} \cdot \boldsymbol{E}^* + \frac{1}{\mu} \boldsymbol{B} \cdot \boldsymbol{B}^* \right) = \frac{\epsilon}{2} |E_0|^2$$

We can also write this as (J 6.106, Pg 259):

$$u = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H})$$

Electromagnetic momentum (J Pg 260-261) The mechanical momentum is (LN 8.36, Pg 472):

$$\vec{p}_{mech} = \rho_m \vec{v}$$

where ρ_m is the mass density and $\vec{v}$ is the velocity field. The rate of change of mechanical momentum (associated with particles) is (J 6.114, Pg 260):

$$\frac{d\boldsymbol{P}_{mech}}{dt} = \int_V (\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}) d^3x$$

because the force per unit volume is (G 8.14, Pg 362):

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$$

The electromagnetic momentum density is (J 6.118, Pg 261):

$$\boldsymbol{g} = \frac{1}{c^2}(\boldsymbol{E} \times \boldsymbol{H}) = \frac{\boldsymbol{S}}{c^2} = \epsilon_0(\boldsymbol{E} \times \boldsymbol{B})$$

so that (J 6.117, Pg 261):

$$\boldsymbol{P}_{field} = \mu_0\epsilon_0 \int_V \boldsymbol{E} \times \boldsymbol{H} d^3x = \int_V \boldsymbol{g} d^3x$$

The local conservation law is (G 8.30, Pg 367), without any changes in mechanical momentum:

$$\frac{\partial \boldsymbol{g}}{\partial t} = \nabla \cdot \underline{\underline{T}}$$

Angular momentum density (G 371)

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{g} = \epsilon_0[\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})]$$

Maxwell Stress Tensor (J 6.120, Pg 261)

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{\delta_{\alpha\beta}}{2}(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2) \right]$$

so that (J 6.122, Pg 261):

$$\left[\frac{d(\boldsymbol{P}_{mech} + \boldsymbol{P}_{field})}{dt} \right]_\alpha = \oint_S \sum_\beta T_{\alpha\beta} n_\beta da$$
$$\left[\frac{d(\boldsymbol{P}_{mech} + \boldsymbol{P}_{field})}{dt} \right]_\alpha = \oint_S da\hat{n} \cdot \underline{\underline{T}}$$
$$\iff \frac{\partial}{\partial t}(\vec{p}_{mech} + \vec{g}) = \nabla \cdot \underline{\underline{T}}$$

where $\boldsymbol{n}$ is the outward normal to S.
Using Griffith notation:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij} E^2}{2} \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{\delta_{ij} B^2}{2} \right)$$

For a plane wave moving in the $\hat{z}$ direction with polarization in $\hat{x}$, only the $\hat{z}\hat{z}$

term (i.e. T_{33}) is non-zero (LN 9.34-35, Pg 498-499):

$$\underline{\underline{T}}_{plane} = -u(t)\hat{z}\hat{z}$$
$$\underline{\underline{T}}_{plane} = -\hat{k}\hat{k}\epsilon E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)$$

The force per unit volume, mechanical momentum per unit volume, and total EM force can be written as (G 8.19-20, Pg 363):

$$\boldsymbol{f} = \frac{\partial \vec{p}_{mech}}{\partial t} = \nabla\cdot\underline{\underline{T}} - \epsilon_0\mu_0\frac{\partial \boldsymbol{S}}{\partial t}$$
$$\boldsymbol{F} = \oint_S \underline{\underline{T}}\cdot d\boldsymbol{a} - \epsilon_0\mu_0\frac{d}{dt}\int_V \boldsymbol{S}d\tau$$

Tensor operations (LN 8.33, Pg 470)

$$\boldsymbol{a}\cdot\underline{\underline{T}} = \sum_i a_i T_{ij}\hat{r}_j$$
$$\underline{\underline{T}}\cdot\boldsymbol{a} = \sum_j \hat{r}_i T_{ij} a_j$$
$$\nabla\cdot\underline{\underline{T}} = \sum_i \hat{r}_j \frac{\partial}{\partial r_i} T_{ij}$$

Radiation pressure (LN 9.36, Pg 499) for a perfect absorber

$$P = u(t)$$

General polarization state (J 7.19, Pg 299)

$$\boldsymbol{E}(\boldsymbol{x},t) = (\boldsymbol{\epsilon}_1 E_1 + \boldsymbol{\epsilon}_2 E_2)e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$$

Circular polarization (J Pg 300) Define the complex orthogonal unit vectors

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)$$

satisfying orthonormality relations:

$$\boldsymbol{\epsilon}_\pm^*\cdot\boldsymbol{\epsilon}_\mp = 0$$
$$\boldsymbol{\epsilon}_\pm^*\cdot\boldsymbol{\epsilon}_3 = 0$$
$$\boldsymbol{\epsilon}_\pm^*\cdot\boldsymbol{\epsilon}_\pm = 1$$

so that the general polarization state can be written as a superposition of circular polarization states:

$$\boldsymbol{E}(\boldsymbol{x},t) = (E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-)e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$$

Let $\frac{E_-}{E_+} = re^{i\alpha}$. Then the polarization ellipse axes satisfy:

$$\frac{a}{b} = \left|\frac{1+r}{1-r}\right|$$

and the electric field ellipse is rotated anticlockwise (looking into the wave) from $\boldsymbol{\epsilon}_1$ by angle $\alpha/2$.

EM waves at interfaces Snell's law:

$$n_1\sin\theta_i = n_2\sin\theta_t$$

as a consequence of LN Pg 513:

$$\hat{s}\cdot\vec{k}_i = \hat{s}\cdot\vec{k}_r = \hat{s}\cdot\vec{k}_t$$

where $\hat{s}$ is a tangential unit vector.

Fresnel equations (LN Pg 518-520) Define:

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i} > 0$$
$$\beta = \frac{Z_1}{Z_2} > 0$$

At normal incidence, $\alpha = 1$. If $\mu_1 = \mu_2$, then:

$$\beta = \frac{n_2}{n_1}$$

For TM (E parallel to plane):

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \frac{\alpha-\beta}{\alpha+\beta}$$
$$\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \frac{2}{\alpha+\beta}$$

For TE (E perpendicular to plane):

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \frac{1-\alpha\beta}{1+\alpha\beta}$$
$$\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \frac{2}{1+\alpha\beta}$$

Power reflection coefficient (LN Pg 529)

$$I_j = \frac{1}{2}\frac{c}{Z_j}E_j^2\cos\theta_j$$

Hence we need to multiply by the impedances and cosines to get the power ratios:

$$R = \left(\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}}\right)^2$$
$$R = \begin{cases}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2, & \parallel \\ \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2, & \perp\end{cases}$$
$$T = \frac{Z_1}{Z_2}\left(\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}}\right)^2\frac{\cos\theta_t}{\cos\theta_i}$$
$$T = \begin{cases}\alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2, & \parallel \\ \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2, & \perp\end{cases}$$

Brewster's angle (LN 9.104, Pg 525) for $\mu_1 = \mu_2$.

$$\tan\theta_B = \frac{n_2}{n_1}$$

Otherwise, solve for the case where the plane-polarized coefficient vanishes.

Wave equation in conducting media (LN 9.122-123, Pg 533)

$$\nabla^2\boldsymbol{E} = \epsilon\mu\frac{\partial^2\boldsymbol{E}}{\partial t^2} + \sigma\mu\frac{\partial\boldsymbol{E}}{\partial t}$$
$$\nabla^2\boldsymbol{B} = \epsilon\mu\frac{\partial^2\boldsymbol{B}}{\partial t^2} + \sigma\mu\frac{\partial\boldsymbol{B}}{\partial t}$$

Conducting matter wavenumber (LN 9.124-126, Pg 534)

$$\vec{k}\cdot\vec{k} = \epsilon\mu\omega^2 + i\sigma\mu\omega$$
$$|\vec{k}| = k + i\kappa$$

See LN 534 for the full expression.

Good and bad conductors (LN Pg 541)

$$\text{Poor: } \frac{\sigma}{\epsilon\omega} \ll 1$$
$$\text{Good: } \frac{\sigma}{\epsilon\omega} \gg 1$$

Conductor boundary conditions (LN 9.159-160, Pg 546)

$$\hat{n}\cdot[\epsilon_1\boldsymbol{E}_1 - \epsilon_2\boldsymbol{E}_2] = \sigma_f$$
$$\hat{n}\cdot[\boldsymbol{B}_1 - \boldsymbol{B}_2] = 0$$
$$\hat{s}\cdot[\boldsymbol{E}_1 - \boldsymbol{E}_2] = 0$$
$$\hat{s}\cdot\left[\frac{\boldsymbol{B}_1}{\mu_1} - \frac{\boldsymbol{B}_2}{\mu_2}\right] = (\boldsymbol{K}_f\times\hat{n})\cdot\hat{s}$$

Square rooting Suppose $\vec{k}\cdot\vec{k} = \Re + i\Im$. Then

$$\sqrt{\vec{k}\cdot\vec{k}} = k + i\kappa$$
$$k = \sqrt{\Re}\sqrt{\frac{\sqrt{1+\frac{\Im^2}{\Re^2}}+1}{2}}$$
$$\kappa = \sqrt{\Re}\sqrt{\frac{\sqrt{1+\frac{\Im^2}{\Re^2}}-1}{2}}$$

Plasma frequency (LN 9.203, Pg 561)

$$\omega_p^2 = \frac{NZq^2}{m\epsilon_0}$$

where Z is the number of electrons per site.

Two-conductor transmission line (HM 7.2-7.4, Pg 226-227) For a resistanceless line,

$$\frac{\partial \Delta v}{\partial t} = -\frac{1}{\mathcal{C}}\frac{\partial I}{\partial z}$$
$$\frac{\partial I}{\partial t} = -\frac{1}{\mathcal{L}}\frac{\partial \Delta v}{\partial z}$$
$$\implies \frac{\partial^2 \Delta v}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 v}{\partial t^2}$$

The characteristic impedance is (HM 7.11, Pg 228):

$$Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{\Delta v(z,t)}{I(z,t)}$$

The current is in phase with the voltage wave for this resistanceless line. For a mismatched output load impedance at $z = l$, use the ansatz:

$$v(z,t) = V_+ e^{i(kz-\omega t)} + V_- e^{i(-kz-\omega t)}$$
$$i(z,t) = I_+ e^{i(kz-\omega t)} + I_- e^{i(-kz-\omega t)}$$

the input (generator) impedance is (HM 7.17, Pg 229):

$$Z_{gen} = \frac{v(0,t)}{i(0,t)} = Z_0 \frac{Z_{load} - iZ_0 \tan kl}{Z_0 - iZ_{load}\tan kl}$$

Note that when $Z_{load} = Z_0$, then $Z_{gen} = Z_0$. The amplitude reflection coefficient r and power reflection coefficient $\mathcal{R}$ are (HM 7.18-19, Pg 229):

$$r = \frac{V_- e^{-ikl}}{V_+ e^{ikl}} = \frac{Z_{load} - Z_0}{Z_{load} + Z_0}$$
$$\mathcal{R} = \left|\frac{Z_{load} - Z_0}{Z_{load} + Z_0}\right|^2$$

Transmission line properties (LN 9.216, Pg 572)

$$v = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\epsilon\mu}}$$

Transmission line junction reflection and transmission (LN 9.242-243, Pg 583) Amplitude coefficients:

$$\tilde{r} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$
$$\tilde{t} = \frac{2Z_2}{Z_1 + Z_2}$$

Power coefficients:

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2$$
$$T = \frac{Z_1}{Z_2}\left(\frac{2Z_2}{Z_1 + Z_2}\right)^2$$

Transmission line power (LN 9.237, Pg 581)

$$P = \frac{1}{2}\Re(I_0^* V_0) = \frac{1}{2}\frac{|V_0|^2}{Z_{LC}}$$

Waveguide nomenclature (J 8.2) Write the electric field in its parallel and transverse components:

$$\boldsymbol{E}_z = \hat{z}E_z$$
$$\boldsymbol{E}_t = (\hat{z} \times \boldsymbol{E}) \times \hat{z}$$

Conducting plane waveguide (HM 231-234) with free space inside. The TE wave electric field is given by:

$$\boldsymbol{E}_0 = \hat{x}E_0^0 e^{-i\omega t} e^{i(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega t)}$$
$$= \hat{x}E_0^0 e^{-i\omega t} e^{ik_0(-y\cos\theta_0 + z\sin\theta_0)}$$

where θ_0 is the angle from the normal, and the wave is taken to propagate in the negative $\hat{y}$ and positive $\hat{z}$ direction. If the conducting planes are b apart, the quantization condition to make the electric field vanish at each plate it:

$$k_0 b\cos\theta_0 = n\pi, \quad n = 1, 2, 3, \ldots$$

and the cut off frequency for the nth mode is:

$$\lambda_c = \frac{2b}{n}$$

The effective wavelength in the $\hat{z}$ direction of propagation is called the guide wavelength:

$$\lambda_g = \frac{\lambda_0}{\sin\theta_0}$$

where $\lambda_0 = \frac{2\pi}{k_0} = 2\pi\frac{\omega}{c}$ is the free-space wavelength. The wavenumber relations satisfy:

$$k_0^2 = k_c^2 + k_g^2$$

The velocities of the wave are:

$$u_p = \frac{c}{\sin\theta_0} > c$$
$$u_g = c\sin\theta_0 < c$$
$$u_p u_g = c^2$$

In general (J 8.54, Pg 364):

$$v_p v_g = \frac{1}{\mu\epsilon}$$

because $\omega\Delta\omega \propto k\Delta k$.

Hollow conductor waveguide (HM 235-238) Define the transverse Laplacian operator:

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

Consider the complex ansatz inside the hollow region:

$$\boldsymbol{E} = \boldsymbol{E}_0(x,y)e^{i(k_g z - \omega t)}$$
$$\boldsymbol{B} = \boldsymbol{B}_0(x,y)e^{i(k_g z - \omega t)}$$

Then Maxwell's equations are equivalent to the two-dimensional Helmholtz equation:

$$(\nabla_t^2 + k_c^2)\boldsymbol{E}_0 = 0$$
$$(\nabla_t^2 + k_c^2)\boldsymbol{B}_0 = 0$$

where $k_c^2 = \gamma^2 = k_{\epsilon\mu}^2 - k^2 = \omega^2\epsilon\mu - k^2$ (LN 9.268, Pg 592). $k_{\epsilon\mu}$ (or k_0) is the unconfined wavenumber, while k (or k_g) is the wavenumber for propagation on the z direction.

The transverse components of the fields can be obtained from the longitudinal components (holds for all modes):

$$E_x^0 = \frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial y} + k_g\frac{\partial E_z^0}{\partial x}\right)$$
$$E_y^0 = -\frac{i}{k_c^2}\left(k_0\frac{\partial B_z^0}{\partial x} - k_g\frac{\partial E_z^0}{\partial y}\right)$$
$$B_x^0 = -\frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial y} - k_g\frac{\partial B_z^0}{\partial x}\right)$$
$$B_y^0 = \frac{i}{k_c^2}\left(k_0\frac{\partial E_z^0}{\partial x} + k_g\frac{\partial B_z^0}{\partial y}\right)$$

These can be combined to give (J 8.26ab, Pg 358):

$$\boldsymbol{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[k\nabla_t E_z - \omega\hat{z} \times \nabla_t B_z\right]$$
$$\boldsymbol{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[k\nabla_t B_z + \mu\epsilon\omega\hat{z} \times \nabla_t E_z\right]$$

Types of modes (HM 238-240)
TE: $E_z^0 = 0, B_z^0 \neq 0$ and:

$$\nabla B_z^0 = -\frac{ik_c^2}{k_g}\boldsymbol{B}_{t0}$$
$$\boldsymbol{B}_{t0} = \frac{k_g}{k_0}(\hat{z} \times \boldsymbol{E}_{t0})$$
$$\boldsymbol{H}_{t0} = \frac{1}{Z_{\epsilon\mu}}\frac{k_g}{k_0}(\hat{z} \times \boldsymbol{E}_{t0})$$
$$(\nabla_t^2 + k_c^2)B_z^0 = 0$$

TM: $B_z^0 = 0, E_z^0 \neq 0$ and:

$$\nabla_\perp E_z^0 = -\frac{ik_c^2}{k_g}\boldsymbol{E}_{t0}$$
$$\boldsymbol{E}_{t0} = -\frac{k_g}{k_0}(\hat{z}\times\boldsymbol{B}_{t0})$$
$$\boldsymbol{H}_{0t} = \frac{1}{Z_{\epsilon\mu}}\frac{k_{\epsilon\mu}}{k}\hat{z}\times\boldsymbol{E}_{0t}$$
$$(\nabla_t^2 + k_c^2)E_z^0 = 0$$

Both TE and TM (J 8.31-32, Pg 359):

$$\boldsymbol{H}_t = \pm\frac{1}{Z}\hat{z}\times\boldsymbol{E}_t$$
$$Z = \begin{cases} \frac{k}{\epsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}}, & TM \\ \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}}, & TE \end{cases}$$

where the $\pm$ comes from the sign in $e^{\pm ikz}$.

TEM modes satisfy: (J Pg 358)

$$\nabla_t\times\boldsymbol{E}_t = 0$$
$$\nabla_t\cdot\boldsymbol{E}_t = 0$$

with dispersion relation:

$$k = k_0 = \omega\sqrt{\mu\epsilon}$$

and field relation:

$$\boldsymbol{B}_t = \pm\sqrt{\mu\epsilon}\hat{z}\times\boldsymbol{E}_t$$
$$\boldsymbol{H}_{0t} = \hat{z}\times\boldsymbol{E}_{0t}\frac{1}{Z_{\epsilon\mu}}$$

Alternative formulation of Modes from Jackson (J Pg 360) For **TM**:

$$E_z = \psi e^{\pm ikz}$$
$$\boldsymbol{E}_t = \pm\frac{ik}{\gamma^2}\nabla_t\psi$$

For **TE**:

$$H_z = \psi e^{\pm ikz}$$
$$\boldsymbol{H}_t = \pm\frac{ik}{\gamma^2}\nabla_t\psi$$

where γ and ψ satisfy:

$$\gamma^2 = \mu\epsilon\omega^2 - k^2 \geq 0$$
$$(\nabla_t^2 + \gamma^2)\psi = 0$$

with boundary condition:

$$\psi|_S = 0, \quad TM$$
$$\left.\frac{\partial\psi}{\partial n}\right|_S = 0, \quad TE$$

$\gamma_\lambda, \lambda = 1, 2, 3\ldots$ is quantized, and this gives the wavenumbers for each value of λ:

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$$

The frequency when $k_\lambda = 0$ is the cutoff frequency:

$$\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}}$$
$$\implies k_\lambda = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2}$$

The last equation above is the dispersion relation.

Waveguide boundary conditions (HM 7.65-66, Pg 239)

$$\hat{n}\times\boldsymbol{E}|_S = 0$$
$$\hat{n}\cdot\boldsymbol{B}|_S = 0$$

or:

$$\left.\frac{\partial B_z^0}{\partial n}\right|_S = 0, \quad \text{useful for TE}$$
$$E_z^0\big|_S = 0, \quad \text{useful for TM}$$

TE and TM dispersion relation (LN 9.276, Pg 601) and cutoff frequency

$$\omega^2 = v_{\epsilon\mu}^2(k^2 + \gamma_n^2)$$
$$\omega_{c,n} = v_{\epsilon\mu}\gamma_n$$

so that the propagation constant is:

$$k_n(\omega) = k_{\epsilon\mu}\sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}}$$

Rectangular waveguides (HM 240-245)

$$B_z^0 = B^0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \quad \text{TE}$$
$$E_z^0 = E^0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \text{TM}$$

(these are the ψ scalar fields in Jackson) with cutoff frequency:

$$\omega_{mn} = ck_c = \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Note that the lowest mode in TE is TE_{10} while the lowest mode in TM is TM_{11}.

EM fields in Finite conductivity media (J S8.1)

$$\boldsymbol{H}_c \approx \boldsymbol{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta}$$
$$\boldsymbol{E}_c \approx \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)\hat{n}\times\boldsymbol{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta}$$

where $\boldsymbol{E}_c, \boldsymbol{B}_c$ are the fields inside the conductor, ξ is the normal coordinate into the conductor, and δ is the skin depth:

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}}$$

$\boldsymbol{H}_\parallel$ is the tangential magnetic field outside the surface, which is continuous across the interface because the boundary conditions are:

$$\hat{n}\times(\boldsymbol{H} - \boldsymbol{H}_c) = 0$$
$$\hat{n}\times(\boldsymbol{E} - \boldsymbol{E}_c) = 0$$
$$\hat{n}\cdot(\boldsymbol{B} - \boldsymbol{B}_c) = 0$$

for unit normal $\hat{n}$ pointing outward from the perfect conductor into a perfect nonconductor.

The tangential electric field outside the conductor is:

$$\boldsymbol{E}_\parallel \approx \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\hat{n}\times\boldsymbol{H}_\parallel)$$

and the current density near the surface is:

$$\boldsymbol{J} = \sigma\boldsymbol{E}_c = \frac{1}{\delta}(1-i)(\hat{n}\times\boldsymbol{H}_\parallel)e^{-\xi(1-i)/\delta}$$

which gives an equivalent surface current density:

$$\boldsymbol{K}_{eff} = \int_0^\infty \boldsymbol{J}d\xi = \boldsymbol{n}\times\boldsymbol{H}_\parallel$$

There is a power flow into the conductor since the Poynting vector is nonzero at the surface. The power loss per unit area is:

$$\frac{dP}{da} = -\frac{1}{2}\Re\left[\hat{n}\cdot\boldsymbol{E}\times\boldsymbol{H}^*\right]$$
$$= \frac{\mu\omega\delta}{4}|\boldsymbol{H}_\parallel|^2$$
$$= \frac{1}{2\sigma\delta}|\boldsymbol{K}_{eff}|^2$$

Integrating the power loss along the circumference of the waveguide gives the power loss per unit transmitted distance (J 8.58, Pg 364):

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C|\boldsymbol{n}\times\boldsymbol{H}|^2dl$$

Transmitted power (J 8.51, Pg 363)

$$P_{TM} = \frac{\epsilon}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}\int_A\psi^*\psi da$$
$$P_{TE} = \frac{\mu}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}\int_A\psi^*\psi da$$

where we integrate over the cross-sectional area of the waveguide.

Waveguide mode power (LN 9.306-307, Pg 617) Cross-terms between modes vanish when integrated over the cross section. Hence the power is just the weighted average of individual mode powers:

$$\langle P\rangle = \sum_m |c_m^{TM}|^2\langle P_m^{TE}\rangle + \sum_n |c_n^{TE}|^2\langle P_n^{TE}\rangle$$

The same relation holds for energy density U.

Energy density of waveguide (J 8.52, Pg 364)

$$U_{TM} = \frac{\epsilon}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 \int_A \psi^*\psi da$$
$$U_{TE} = \frac{\mu}{2}\left(\frac{\omega}{\omega_\lambda}\right)^2 \int_A \psi^*\psi da$$

Waveguide mode group velocity (LN 9.130, Pg 618)

$$v_{g,n} = \frac{d\omega}{dk_n(\omega)} = v_{\epsilon\mu}\sqrt{1-\frac{\omega_{c,n}^2}{\omega^2}}$$

The group velocity relates the power and energy density:

$$\langle P_n\rangle = v_{g,n}(\omega)\langle U_n\rangle$$

Gauge transformation (LN 10.16, Pg 637)

$$\boldsymbol{A}\to\boldsymbol{A}+\nabla\lambda$$
$$V\to V-\frac{\partial\lambda}{\partial t}$$

Retarded potentials (LN 10.46, Pg 648)

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int_V d\tau'\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}$$
$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int_V d\tau'\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

where the retarded time is:

$$t_r = t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$$

Four-vector transformation (J 11.61-65)

$$(A')^\alpha = \frac{\partial(x')^\alpha}{\partial x^\beta}A^\beta$$
$$(B')_\alpha = \frac{\partial x^\beta}{\partial(x')^\alpha}B_\beta$$
$$(F')^{\alpha\beta} = \frac{\partial(x')^\alpha}{\partial x^\gamma}\frac{\partial(x')^\beta}{\partial x^\delta}F^{\gamma\delta}$$
$$(G')_{\alpha\beta} = \frac{\partial x^\gamma}{\partial(x')^\alpha}\frac{\partial x^\delta}{\partial(x')^\beta}G_{\gamma\delta}$$
$$(H')^\alpha_\beta = \frac{\partial(x')^\alpha}{\partial x^\gamma}\frac{\partial x^\delta}{\partial(x')^\beta}H^\gamma_\delta$$

Four vector dot product

$$B\cdot A = B_\alpha A^\alpha$$

Four-vectors (LN)

$$\partial_\mu = \left(\frac{\partial}{\partial r^0},\frac{\partial}{\partial r^1},\frac{\partial}{\partial r^2},\frac{\partial}{\partial r^3}\right)$$
$$\Box^2 = \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$
$$v^\mu = \gamma(c,\vec{v})$$
$$J^\mu = (\rho c, \rho\vec{v})$$
$$A^\mu = \left(\frac{V}{c},\vec{A}\right)$$
$$F^\mu = \gamma\left(\frac{\vec{F}\cdot\vec{v}}{c},\vec{F}\right)$$

Kronecker Delta

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$
$$g_{\alpha\gamma}g^{\gamma\beta} = \delta^\beta_\alpha$$

Lorentz transform In matrix form:

$$\Lambda^T g\Lambda = g$$

where g is the metric tensor. In tensor notation (LN 12.36, Pg 783):

$$\tilde{F}^{\mu\nu} = \Lambda^\mu_\lambda\Lambda^\nu_\sigma F^{\lambda\sigma}$$

Field tensors and components (LN Pg 783-784)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$F^{j0} = -F^{0j} = \frac{E_j}{c}$$
$$F^{ij} = -\epsilon_{ijk}B_k$$
$$\beta_i\beta_j F^{ij} = -\beta_i\beta_j\epsilon_{ijk}B_k = 0$$
$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$
$$G^{00} = F^{00} = 0$$
$$G^{jj} = F^{jj} = 0$$
$$G^{j0} = -G^{0j} = B_j$$
$$G^{ij} = \frac{1}{c}\epsilon_{ijk}E_k$$

Transformation of fields (LN 12.54, Pg 787)

$$\tilde{E}_\parallel = E_\parallel$$
$$\tilde{B}_\parallel = B_\parallel$$
$$\tilde{\vec{E}}_\perp = \gamma[\vec{E}_\perp - \vec{v}\times\vec{B}_\perp]$$
$$\tilde{\vec{B}}_\perp = \gamma\left[\vec{B}_\perp + \frac{1}{c^2}\vec{v}\times\vec{E}_\perp\right]$$

where i,j,k,l only run over 1,2,3.

Tensor identities (LN)

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$$
$$\epsilon_{jkl}\beta_k\beta_l = 0$$

Key four-vector equations (LN)

$$\text{Charge conservation: } \partial_\mu J^\mu = 0$$
$$\text{Wave equation: } \Box^2\overline{\overline{A}} = \mu_0\overline{\overline{J}}$$
$$\text{Lorenz gauge: } \partial_\mu A^\mu = 0$$
$$-\frac{c^2}{2}F_{\mu\nu}F^{\mu\nu} = E^2 - c^2B^2$$
$$-\frac{c}{4}F_{\mu\nu}G^{\mu\nu} = \vec{E}\cdot\vec{B}$$
$$\text{Force: } \frac{dp^\mu}{d\tau} = qF^{\mu\nu}v_\nu$$

Maxwell's equations (LN 12.80, 82, Pg 794)

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$
$$\partial_\mu G^{\mu\nu} = 0$$