Ph12b Book Notes

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Chapter 1

General Notes

1. Rayleigh-Jeans approximation: $u(\nu) = \frac{8\pi\nu^2}{c^3}k_B T$. Where $u(\nu)$ is the energy per frequency interval per unit volume. Total energy per unit volume is $U = \int_0^\infty u(\nu)d\nu$.
2. Planck's equation:
$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_B T} - 1}$$
3. Bohr radius: $r_n = \frac{n^2\hbar^2}{me^2}$.
4. Rydberg constant: $\frac{me^4}{2\hbar^2} = 13.6eV$.
5. Schrodinger's Equation (Time-dependent):
$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$
6. Gaussian (square of the Gaussian wave function):
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$
7. Solution to initial value problem:
$$\psi(\vec{r},t) = \exp\left(\frac{-it\hat{H}}{\hbar}\right)\psi(\vec{r},0) = \hat{U}\psi(\vec{r},0)$$
Note that the eigenvalue equation for this unitary operator is:
$$\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\psi_n = \exp\left(\frac{-iE_n t}{\hbar}\right)\psi_n$$
8. Position displacement operator:
$$\hat{\mathcal{D}}[f(x)] = \exp\left(\frac{idx\hat{p}_x}{\hbar}\right)f(x) = f(x+dx)$$
9. Angular displacement operator:
$$\hat{R}_{\Delta\phi} = \exp\left(\frac{i\vec{\Delta\phi}\cdot\hat{L}}{\hbar}\right)$$
where $\hat{L}$ is the vector operator for the total angular momentum of the system.
10. Schrodinger's Equation (time-independent)
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

11. Expectation Value

$$\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)P(j)$$

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \Psi^* Q(x,p)\Psi dx$$

12. Standard Deviation

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$$

$$\sigma^2 = \left\langle (\hat{Q} - \langle Q\rangle)^2 \right\rangle$$

13. Momentum

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi dx = m\frac{d\langle x\rangle}{dt}$$

14. Position

$$\langle x\rangle = \int_{-\infty}^{\infty} \Phi^* \left(\frac{-\hbar}{i}\frac{\partial}{\partial p}\right)\Phi dp$$

where $\Phi(p,t)$ is the momentum space wave function.

15. Velocity:

$$v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{p}{2m} = \frac{v_{cl}}{2}$$

$$v_g = \frac{\partial\omega}{\partial k} = \frac{p}{m} = v_{cl}$$

16. Position and Momentum space correspondence:

$$\text{Position space: } \hat{Q}(x,p) \to \hat{Q}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

$$\text{k-space: } \hat{Q}(x,p) \to \hat{Q}\left(-i\frac{\partial}{\partial k}, \hbar k\right)$$

$$\text{Momentum space: } \hat{Q}(x,p) \to \left(\frac{-\hbar}{i}\frac{\partial}{\partial p}, p\right)$$

17. Hamiltonian Operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

18. Hamilton's Equations:

$$\frac{\partial H}{\partial x} = -\dot{p}_x \quad \frac{\partial H}{\partial p_x} = \dot{x}$$

$$\frac{\partial H}{\partial y} = -\dot{p}_y \quad \frac{\partial H}{\partial p_y} = \dot{y}$$

$$\frac{\partial H}{\partial z} = -\dot{p}_z \quad \frac{\partial H}{\partial p_z} = \dot{z}$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta}$$

$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r \quad \frac{\partial H}{\partial p_r} = \dot{r}$$

In general:

$$\frac{\partial H}{\partial q_l} = -\dot{p}_l, \quad \frac{\partial H}{\partial p_l} = \dot{q}_l, \quad l = 1, 2, \ldots, N$$

19. General solution to TISE:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

20. Infinite square well: Boundary conditions: $\psi(0) = \psi(a) = 0$. Energy: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{\hbar^2 k_n^2}{2m}, k_n = \frac{n\pi}{a}$. One solution: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$.

21. Orthonormality: $\int \psi_m(x)^* \psi_n(x) dx = \delta_{mn}$.

22. Delta function orthogonality:

$$\langle \psi_k | \psi_{k'} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(k'-k)} dx = \delta(k'-k)$$

23. Fourier decomposition. Given $f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$, obtain $c_n = \int \psi_n(x)^* f(x) dx$.

24. General solution to stationary states:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$
$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0) dx$$

25. Energy expectation value: $\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$.

26. TISE for harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \implies \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]\psi = E\psi$$

27. Canonical commutation relation: $[x,p] = i\hbar$

28. Ground state of harmonic oscillator:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/3} e^{-\frac{m\omega}{2\hbar}x^2}, E_0 = \frac{1}{2}\hbar\omega$$

29. Energy expectation values:

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m}$$
$$\langle V \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

30. Normalised stationary states for harmonic oscillator:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}, \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

31. Hermite polynomials:

$$H_0 = 1$$
$$H_1 = 2\xi$$
$$H_2 = 4\xi^2 - 2$$
$$\text{Rodrigues formula: } H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2} \implies H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$$
$$\frac{dH_n}{d\xi} = 2nH_{n-1}(\xi)$$
$$\text{Generating function: } e^{-z^2+2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi)$$

32. Plancherel's theorem:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \iff F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Say that $F(k)$ is the Fourier transform of $f(x)$ and that $f(x)$ is the inverse Fourier transform of $F(k)$.

33. The Fourier transform of a Gaussian is also a Gaussian. A Gaussian in space with width Δx will be a Gaussian in wavenumber with width $\Delta k = \frac{1}{2\Delta x}$.

34. Wave packet solution to Free Particle system:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \iff \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$$

35. Delta function well bound state: Given $V(x) = -\alpha\delta(x), \alpha > 0$,

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}, E = -\frac{m\alpha^2}{2\hbar^2}$$

36. Reflection and Transmission coefficients (relative probability that a particle will be reflected/transmitted):

$$R = \frac{1}{1 + 2\hbar^2 E/m\alpha^2}$$
$$T = \frac{1}{1 + m\alpha^2/2\hbar^2 E}$$

37. Strategies for solving potential distributions:
    (a) Make solution guess in each region (complex exponentials or trigo functions)
    (b) Impose boundary conditions (wavefunction vanishes at infinity, continuity of wavefunction across boundaries, continuity of derivative across boundaries), no incoming wave from the other side for scattering states)
    (c) Normalisation (or infinite superposition for scattering states)

38. Hilbert Space: A (real or complex) complete inner product space. Complete: every Cauchy sequence of functions converges to a function that is also in the space.

39. Inner product of two functions: $\langle f|g\rangle = \int_a^b f(x)^* g(x) dx$.

40. Conjugate of inner product (skew symmetry): $\langle g|f\rangle = \langle f|g\rangle^*$

41. Inner product is positive semidefinite: $\langle f|f\rangle \geq 0, \langle f|f\rangle = 0 \iff f(x) = 0$.

42. Inner product is antilinear in first element: $\langle ax_1 + bx_2|y\rangle = a^*\langle x_1|y\rangle + b^*\langle x_2|y\rangle$.

43. Inner product is linear in the second element: $\langle x|ay_1 + by_2\rangle = a\langle x|y_1\rangle + a\langle x|y_2\rangle$.

44. Integral Schwarz inequality: $\left|\int_a^b f(x)^* g(x) dx\right| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$.

45. Expectation value: $\langle Q\rangle = \int \Psi^* \hat{Q}\Psi dx = \langle\Psi|\hat{Q}\Psi\rangle = \sum_i \langle\Psi|\omega_i\rangle\langle\omega_i|\Psi\rangle\omega_i = \sum_i |\langle\omega_i|\Psi\rangle|^2\omega_i$.

46. Operators representing observables must give a real expectation value $\langle Q\rangle = \langle Q\rangle^*$ and hence must be Hermitian: $\langle f|\hat{Q}f\rangle = \langle\hat{Q}f|f\rangle, \forall f(x)$.

47. The product of two Hermitian operators is Hermitian iff they commute.

48. If Ω and Λ are two commuting Hermitian operators, there exists a basis of common eigenvectors that diagonalises them both.

49. Matrix Representation of Operators:

$$\Omega_{ji} = \langle j|\Omega|i\rangle$$
$$\Omega|V\rangle = \Omega\sum_i v_i|i\rangle = \sum_i v_i\Omega|i\rangle = \sum_i v'_i|i\rangle = |V'\rangle$$
$$v'_i = \sum_j \Omega_{ij}v_j$$
$$\begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{pmatrix} = \begin{pmatrix} \langle 1|\Omega|1\rangle & \langle 1|\Omega|2\rangle & \cdots & \langle 1|\Omega|n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n|\Omega|1\rangle & \langle n|\Omega|2\rangle & \cdots & \langle n|\Omega|n\rangle \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

The columns are the components of the transformed basis vector under Ω in the given basis.

50. Projection Operator:
    - Note the completeness relation $|V\rangle = (\sum_{i=1}^n |i\rangle\langle i|)\,|V\rangle$.
    - Hence $|i\rangle\langle i|$ is the projection operator $\mathbb{P}_i$ for the ket $|i\rangle$.
    - On kets, $\mathbb{P}_i|V\rangle = |i\rangle\langle i|V\rangle = |i\rangle v_i$.
    - On bras: $\langle V|\mathbb{P}_i = \langle V|i\rangle\langle i| = v_i^*\langle i|$.
    - The identity operator: $\mathbb{I} = \sum_{i=1}^n \mathbb{P}_i$. In infinity dimensions, $\mathbb{I} = \int |x'\rangle\langle x'|dx'$.
    - Product: $\mathbb{P}_i\mathbb{P}_j = \delta_{ij}\mathbb{P}_j$.

51. Matrix product (prove by writing the identity in between the operators and noting the identity can be written as a sum of projection operators):

$$(\Omega\Lambda)_{ij} = \langle i|\Omega\Lambda|j\rangle = \sum_k \Omega_{ik}\Lambda_{kj}$$

52. Anticommutator: $\{A, B\} = AB + BA$.

53. Commutation identities:
    - $\hat{A}$ commutes with any function $f(\hat{A})$.

$$[A, BC] = B[A, C] + [A, B]C$$
$$[AB, C] = A[B, C] + [A, C]B$$

- $[A + B, C] = [A, C] + [B, C]$
- $[A, BC] = [A, B]C + B[A, C]$
- $[A, BCD] = [A, B]CD + B[A, C]D + BC[A, D]$
- $[A, BCDE] = [A, B]CDE + B[A, C]DE + BC[A, D]E + BCD[A, E]$
- $[AB, C] = A[B, C] + [A, C]B$
- $[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC$
- $[ABCD, E] = ABC[D, E] + AB[C, E]D + A[B, E]CD + [A, E]BCD$
- $[AB, CD] = A[B, CD] + [A, CD]B = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$
- $[[[A, B], C], D] + [[[B, C], D], A] + [[[C, D], A], B] + [[[D, A], B], C] = [[A, C], [B, D]]$
- $[AB, C] = A\{B, C\} - \{A, C\}B$, where $\{A, B\} = AB + BA$ is the anticommutator defined above.

54. Hermitian conjugate (adjoint): The adjoint of an operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that $\langle f|\hat{Q}g\rangle = \langle \hat{Q}^\dagger f|g\rangle$ for all f and g. A Hermitian operators is its own Hermitian conjugat.

55. General rule for taking adjoint of a product: Reverse the order of all factors, replace everything with the adjoint: $\Omega \leftrightarrow \Omega^\dagger, |V\rangle \leftrightarrow \langle V|, a \leftrightarrow a^*$.

56. The eigenvalues of a Hermitian operator are real. Also, all Hermitian operators are diagonalisable and has an orthonormal eigenbasis.

57. Every operator can be written as a sum of a Hermitian and an Antihermitian operator:

$$\Omega = \frac{\Omega + \Omega^\dagger}{2} + \frac{\Omega - \Omega^\dagger}{2}$$

58. Unitary operator: $UU^\dagger = I$. Unitary operators preserve the inner product: $\langle V_2'|V_1'\rangle = \langle UV_2|UV_1\rangle = \langle V_2|V_1\rangle$. For real vector spaces, we have the orthogonality condition: $U^{-1} = U^T$.

59. Determinate states have $\sigma^2 = 0$, which only occur when $\hat{Q}\Psi = q\Psi$, where q is the value that the measurement returns. Hence q is an eigenvalue of the operator $\hat{Q}$ with eigenfunction Ψ. The zero function cannot be an eigenfunction.

60. Degenerate spectrum: When two or more linearly independent eigenfunctions share the same eigenvalue.

61. Types of eigenfunctions:
    - Discrete eigenvalues: Eigenfunction lies in Hilbert space and constitute physically realizable states. Eigenvalues are real if the operator is Hermitian. Eigenfunctions belonging to distinct eigenvalues are orthogonal $\langle f|g\rangle = 0$.
    - Continuous eigenvalues: Eigenfunctions are not normalizable, but linear combinations may be normalizable.

62. Axiom on completeness: The eigenfunctions of an observable operator are complete: Any function in Hilbert space can be expressed as a linear combination of them.

63. Dirac Delta function from integral: $\delta(p-p') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(p-p')x}dx$.

64. Dirac orthonormality for continuous spectra: $\langle f_{p'}|f_p\rangle = \delta(p-p')$.

65. Completeness of eigenfunctions for continuous spectra: Any function can be written as $f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} c(p)e^{ipx/\hbar}dp$ with the coefficient function calculated using $\langle f_{p'}|f\rangle = \int_{-\infty}^{\infty} c(p)\delta(p-p')dp = c(p')$.

66. Statistical Interpretation:
    - If the particle is in a state $|\psi\rangle$, measurement of the variable corresponding to Ω will yield one of the eigenvalues ω with probability $P(\omega) = \frac{|\langle\omega|\psi\rangle|^2}{\langle\psi|\psi\rangle}$. The denominator is 1 if the state was normalised. The state of the system after the measurement is then the eigenstate $|\omega\rangle$ after the measurement.
    - Discrete spectra: $\Psi(x,t) = \sum_n c_n f_n(x) \implies \langle Q\rangle = \sum_n q_n|c_n|^2, \sum_n |c_n|^2 = 1$.

67. Momentum space wave function: $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)dx$. Note that this is the inverse Fourier transform of the position space wave function. Hence we also have $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\Phi(p,t)dp$.

68. Generalized uncertainty principle: $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$.
    - Commutator of two Hermitian operators is anti-Hermitian: $\hat{Q}^\dagger = -\hat{Q}$ and hence has imaginary expectation value.

69. Commutator identity: $[AB,C] = A[B,C] + [A,C]B$.

70. Minimum uncertainty: The necessary and sufficient condition for minimum uncertainty is that $g(x) = iaf(x), a \in \mathbb{R}$. For position-momentum uncertainty, the general solution is a Gaussian.

71. Common Uncertainty Principles:

$$\Delta x\Delta p \geq \frac{\hbar}{2}$$
$$\Delta t\Delta E \geq \frac{\hbar}{2}$$

    Δt is the amount of time it takes for the expectation value of Q to change by one standard deviation:

$$\sigma_Q = \left|\frac{d\langle Q\rangle}{dt}\right|\Delta t$$

72. Time derivative of operator expectation value:

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$$

    obtained using product rule on LHS and using TDSE. If operator does not depend explicitly on time (typical), third term $\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$ vanishes, and hence if the operator commutes with the Hamiltonian, then $\langle Q\rangle$ is a constant, and hence is a conserved quantity.

73. Special Case: Ehrenfest's Theorem:

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

in 3D:

$$\frac{d}{dt}\langle \mathbf{r}\rangle = \frac{1}{m}\langle \mathbf{p}\rangle$$
$$\frac{d}{dt}\langle \mathbf{p}\rangle = \langle -\nabla V\rangle$$

74. Harmonic Oscillator Operators:

$$\text{Lowering: } \hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right)$$
$$\text{Raising: } \hat{a}^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right)$$
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})$$
$$[\hat{a}, \hat{a}^\dagger] = 1$$
$$\hat{N}\phi_n = \hat{a}^\dagger\hat{\phi}_n = n\phi_n$$

75. Correspondence principle: Classical harmonic oscillator probability density: $P = \frac{1}{\pi\sqrt{x_0^2 - x^2}}$ such that $\int_{-x_0}^{x_0} P(x)dx = 1$ where $\pm x_0$ are the classical turning points.

76. Dirac Notation

- Operators are linear transformations $|\beta\rangle = \hat{Q}|\alpha\rangle$.
- Vectors can be represented in any basis by their components: $|\alpha\rangle = \sum_n a_n|e_n\rangle$ with $a_n = \langle e_n|\alpha\rangle$.
- Operators can be represented by their matrix elements: $\langle e_m|\hat{Q}|e_n\rangle \equiv Q_{mn}$.
  - $\langle x|X|x'\rangle = x\delta(x - x')$
  - $\langle x|P|x'\rangle = -i\hbar\delta'(x - x')$.
- Bra: $\langle f| = \int f^*[\cdots]dx$. In finite-dimensional vector space, it is the Hermitian conjugate of the Ket.
- Projection operator: $\hat{P} = |\alpha\rangle\langle\alpha|$ picks out the portion of any other vector that "lies along" $|\alpha\rangle$.
- Orthonormal basis: $\langle e_m|e_n\rangle = \delta_{mn}$ and the sum of the projects along all the finite dimensions is clearly the identity: $\sum_n |e_n\rangle\langle e_n| = I$.
- Dirac orthonormal continuous basis: $\langle e_z|e_{z'}\rangle = \delta(z - z')$ with analogous integral over the basis: $\int |e_z\rangle\langle e_z|dz = 1$.
- Spectral decomposition: For an operator $\hat{Q}$ with a complete set of orthonormal eigenvectors $\hat{Q}|e_n\rangle = q_n|e_n\rangle, n = 1, 2, 3, \ldots$, we can write the operator as:

$$\hat{Q} = \sum_n q_n|e_n\rangle\langle e_n|$$

77. Virial Theorem:

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{dV}{dx}\right\rangle$$
$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

78. Dealing with the matrix Hamiltonian.

- Find the eigenvalues and eigenvectors of the matrix Hamiltonian.
- Consider the initial conditions (t=0) to find the coefficients of each of the eigenvectors to describe the system.

- Tack on the time-dependence (since the eigenvalues obtained are the energies) to each of the eigenvector terms to obtain the time-evolution of the system.

79. Equations in 3D:
    - Momentum operator: $\mathbf{p} \to \frac{\hbar}{i}\nabla$
    - TDSE: $i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi$
    - TISE: $\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$.
    - Solution to TDSE: $\Psi(\mathbf{r}, t) = \sum_n c_n\psi_n(\mathbf{r})e^{-iE_nt/\hbar}$

80. Canonical commutation relations: Let $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$. Then:

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}$$
$$[r_i, r_j] = [p_i, p_j] = 0$$

81. Spherical coordinates
    - Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2 7Th\theta}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

- Separation of variables: Write $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$.
- Solution for azimuthal angle:

$$\Phi(\phi) = e^{im\phi}, m = 0, \pm1, \pm2, \ldots$$

- Solution for polar angle:

$$\Theta(\theta) = AP_l^m(\cos\theta)$$

- Legendre function:

$$P_l^m(x) \equiv (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_l(x) = P_l^{-m}(x)$$

- Legendre polynomial (using Rodrigues formula):

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

requires that l is a non-negative integer. Note also that if $|m| > l$, then $P_l^m = 0$. Hence for any l there are $2l+1$ possible values of $m : -l, -l+1, \ldots, -1, 0, 1, \ldots, l$.

- Legendre polynomials are orthonormal:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \left(\frac{2}{2l+1}\right)\delta_{ll'}$$

- Call l the azimuthal quantum number and m the magnetic quantum number.
- Normalized angular wavefunction:

$$Y_l^m(\theta, \phi) = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta)$$
$$\epsilon = \begin{cases}(-1)^m, m \geq 0\\ 1, m \leq 0\end{cases}$$

- Radial Equation:

$$u(r) \equiv rR(r)$$
$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

- Effective potential

$$V_{eff} = V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

tends to throw the particle away from the origin.

- General solution to radial equation in infinite spherical well: $u(r) = Arj_l(kr) + Brn_l(kr)$, where $j_l(x)$ is the spherical Bessel function of order l and $n_l(x)$ is the spherical Neumann function of order l:

$$j_l(x) \equiv (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$
$$n_l(x) \equiv -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$$

$B = 0$ because Neumann functions blow up at the origin.

- Energies in infinite spherical well:

$$E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$$

where β_{nl} is the nth zero of the lth spherical Bessel function.

82. Hydrogen Atom

- Bohr formula:

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$

- Bohr radius

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.529 \times 10^{-10} m$$

- Ground state of hydrogen ($n = 1, l = 0, m = 0$):

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

- See page 145 onwards for all the formulae for the hydrogen atom.

83. Angular Momentum operators and commutation relations:

$$\mathbf{L} = \frac{\hbar}{i}(\mathbf{r} \times \nabla)$$
$$L_x = yp_z - zp_y$$
$$L_y = zp_x - xp_z$$
$$L_z = xp_y - yp_x$$
$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$
$$L^2 = L_x^2 + L_y^2 + L_z^2$$
$$[L^2, \mathbf{L}] = 0$$
$$[L_z, L^2] = 0$$
$$[L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, [L_z, z] = 0$$
$$[L_z, p_x] = i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0$$

Note that only one of the three Cartesian components of the angular momentum may be known at one time.

84. Angular momentum stuff in spherical coordinates:

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$
$$\hat{L}_x = i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi} \right)$$
$$\hat{L}_y = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta\sin\phi \frac{\partial}{\partial\phi} \right)$$
$$\hat{L}_z = -i\hbar \frac{\partial}{\partial\phi}$$
$$\hat{L}_+ = \hbar e^{i\phi} \left(i\cot\theta \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta} \right)$$
$$\hat{L}_- = \hbar e^{-i\phi} \left(i\cot\theta \frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta} \right)$$

85. Angular momentum ladder operators (increases or decreases the value of m by 1):

$$L_\pm = L_x \pm iL_y$$
$$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$$
$$[L^2, L_\pm] = 0$$
$$[L_z, L_\pm] = \pm\hbar L_\pm$$
$$L^2 = L_\mp L_\pm + L_z^2 \pm \hbar L_z$$

86. Angular momentum eigenvalue equations:

$$\hat{L}^2\phi_{lm} = \hbar^2 l(l+1)\phi_{lm}, l = 0, 1, 2, \ldots$$
$$\hat{L}\phi_{lm} = \hbar\sqrt{l(l+1)}\phi_{lm}$$
$$\hat{L}_z\phi_{lm} = \hbar m\phi_{lm}, m = -l, \ldots, 0, \ldots, l$$

87. Solid angle $dS = r^2 d\Omega = r^2 \sin\theta d\theta d\phi$.

88. Two body problem: If the interaction potential only depends on the separation between particles $\vec{r} = \vec{r}_1 - \vec{r}_2$, then we define the position of the centre of mass $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ such that $\vec{r}_1 = \vec{R} - \frac{\mu}{m_1}\vec{r}, \vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r}$. Differentiation with respect to particle 1 and 2 positions can then be written as:

$$\nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r$$
$$\nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ is the reduced mass, and the time independent Schrodinger equation becomes:

$$\frac{-\hbar^2}{2(m_1+m_2)}\nabla_R^2\Psi - \frac{\hbar^2}{2\mu}\nabla_r^2\Psi + V(\vec{r})\Psi = E\Psi$$

Alternatively, define the weighted momentum $\mathbf{p} = \frac{m_1\mathbf{p}_2 - m_2\mathbf{p}_1}{m_1+m_2}$ and the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and write the Hamiltonian as:

$$H = \frac{P^2}{2M} + \left[\frac{p^2}{2\mu} + V(r) \right] = H_{CM} + H_{rel}$$

89. Composite wavefunction for indistinguishable particles: $\psi_\pm(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right]$. If the particles are distinguishable, then $\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$. Plus sign for bosons, minus sign for fermions.

90. Symmetrization requirement. For two identical particles, $\psi(\vec{r}_1, \vec{r}_2) = \pm\psi(\vec{r}_2, \vec{r}_1)$, plus sign for bosons, minus sign for fermions.

91. Exchange force: Non-trivial. See page 208.

92. Probability Current Density:

$$\mathbf{J} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$$

with dimensions per area per second.

93. Transmission and Reflection. Given:

$$\psi_{inc} = Ae^{i(k_1 x - \omega_1 t)}$$
$$\psi_{ref} = Be^{i(k_1 x + \omega_1 t)}$$
$$\psi_{trans} = Ce^{i(k_2 x - \omega_2 t)}$$

The reflection and transmission coefficients are:

$$T = \left|\frac{\mathbf{J}_{trans}}{\mathbf{J}_{inc}}\right| = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}$$
$$R = \left|\frac{\mathbf{J}_{refl}}{\mathbf{J}_{inc}}\right| = \left|\frac{B}{A}\right|^2$$

94. Finite rectangular barrier:

$$\frac{1}{T} = 1 + \frac{V^2}{4E(E-V)}\sin^2(2k_2 a), \quad k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}, E > V$$
$$\frac{1}{T} = 1 + \frac{V^2}{4E(E-V)}\sinh^2(2k_2 a), \quad k_2 = \frac{\sqrt{2m(V-E)}}{\hbar}, E < V$$

95. Potential step:

$$T = \frac{4k_2/k_1}{(1+(k_2/k_1))^2}, E > V$$
$$\frac{k_2^2}{k_1^2} = 1 - \frac{V}{E}$$

Transmission is zero if $E < V$, so that the kinetic energy in the entire step region is negative.

96. Scattering off a finite potential well

$$\frac{1}{T} = 1 + \frac{V^2}{4E(E+|V|)}\sin^2(2k_2 a), k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$$

97. Double potential step:

$$T = \frac{4k_1 k_3 k_2^2}{k_2^2(k_1+k_3)^2 + (k_3^2 - k_2^2)(k_1^2 - k_2^2)\sin^2(k_2 a)}, k_1 \geq k_2 \geq k_3$$

98. Delay from reflection and transmission. Let X be the centre of the incident wavepacket at $t = 0$, which moves with group velocity $\frac{\hbar k_0}{m}$.

$$\text{Incident packet: } x = \frac{\hbar k_0}{m}t - X$$
$$\text{Reflected packet: } x = \frac{-\hbar k_0}{m}t + X + \left(\frac{\partial \phi_R}{\partial k}\right)_{k_0}$$
$$\text{Transmitted packet: } x = \frac{\hbar k_0}{m}t - X - \left(\frac{\partial \phi_T}{\partial k}\right)_{k_0}$$

where the wave functions are:

$$\psi_{inc} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(k) e^{ikX} e^{i(kx-\omega t)} dk$$
$$\psi_{refl} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{R} e^{i\phi_R} b(k) e^{ikX} e^{-i(kx+\omega t)} dk$$
$$\psi_{trans} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{T} e^{i\phi_T} b(k) e^{ikX} e^{i(kx-\omega t)} dk$$

99. Plane wave solution to 3D Schrodinger equation (free particle in 3D):

$$\psi_{\mathbf{k}}(\mathbf{r}, t) = A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \quad \hbar\omega = E_k$$

100. 3D Dirac Delta:

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$
$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \iiint e^{i(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'))} d\mathbf{k}$$
$$d\mathbf{k} = dk_x dk_y dk_z$$
$$\psi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} \iff \iiint \psi_{\mathbf{k}}^* \psi_{\mathbf{k}'} dx dy dz = \delta(\mathbf{k} - \mathbf{k}')$$

101. 3D Wavepacket:

$$\psi(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \iiint b(\mathbf{k}, t) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} d\mathbf{k}$$
$$b(\mathbf{k}, t) = \frac{1}{(2\pi)^{3/2}} \iiint \psi(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} d\mathbf{r}$$

102. Radial and Angular momentum:

$$\hat{H} = \frac{p^2}{2m} = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2}, \quad p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{2}\left(\frac{1}{r}\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}\frac{1}{r}\right)$$

p_r is not just simplify $\frac{\mathbf{r}\cdot\mathbf{p}}{r}$ because this is not Hermitian.

103. Central potential radial equation. Write $\phi = R(r)Y_l^m(\theta, \phi)$. Then $R(r)$ satisfies:

$$\left[\frac{p_r^2}{2\mu} + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r)\right] R(r) = ER(r)$$

where the second term is the angular momentum barrier.