Ph125ab Book Notes: Quantum Mechanics

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Schwarz Inequality (1.3.15, Pg 16)

$$|\langle V|W\rangle| \leq |V||W|$$

Triangle inequality (1.3.16, Pg 16)

$$|V+W| \leq |V| + |W|$$

Vector triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

Summation relations

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

Gaussian integral

$$\int_{-\infty}^{\infty} dz e^{-kz^2} = \sqrt{\frac{\pi}{k}}$$

$$\int_{-\infty}^{\infty} dz A e^{az^2+bz+c} = \sqrt{\frac{\pi}{-a}} A e^{c-\frac{b^2}{4a}}$$

In n-dimensions, just bring both sides to the appropriate nth power by separating the integral into a product of independent terms.

Matrix inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Adding subspaces Given to subspaces $\mathbb{V}_i^{n_i}, \mathbb{V}_j^{n_j}$, define the sum $\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{n_j} = \mathbb{V}_k^{n_k}$ to be the set containing all elements of each of the subspaces as well as all possible linear combinations of all the elements of both subspaces:

Commutator Identities (1.5.10-11, Pg 20)

$$[\Omega, \Lambda\Theta] = \Lambda[\Omega,\Theta] + [\Omega,\Lambda]\Theta$$

$$[\Lambda\Omega, \Theta] = \Lambda[\Omega,\Theta] + [\Lambda,\Theta]\Omega$$

Matrix elements of linear operators (1.6.1, Pg 21) Let vectors be expressed as columns. Then the jth row, ith column component of the operator matrix can be written as:

$$\langle j|\Omega|i\rangle = \Omega_{ji}$$

Hermitian/Anti-Hermitian Operator (Definition 13-14, Pg 27)

$$\Omega^\dagger = \Omega \quad \text{Hermitian}$$

$$\Omega^\dagger = -\Omega \quad \text{Anti-Hermitian}$$

$$\Omega = \frac{\Omega+\Omega^\dagger}{2} + \frac{\Omega-\Omega^\dagger}{2}$$

Hermitian operator eigenvalues are real (Theorem 9, Pg 35). Every Hermitian operator has a basis of orthonormal eigenvectors and is diagonal in this eigenbasis with eigenvalues along the diagonal (Theorem 10, Pg 36).

Unitary (Definition 15, Pg 28)

$$UU^\dagger = I$$

Unitary operators preserve the inner product when operating on both bra and ket (Theorem 7, Pg 28). The rows and columns of a unitary matrix are orthonormal (Theorem 8, Pg 28). The eigenvalues are complex numbers of unit modulus (Theorem 11, Pg 39). The eigenvectors of a non-degenerate unitary operator are mutually orthogonal (Theorem 12, Pg 39).

Trace properties (Ex 1.7.1, Pg 30)

$$\text{Tr}(\Omega\Lambda) = \text{Tr}(\Lambda\Omega)$$

$$\text{Tr}(XYZ) = \text{Tr}(YZX) = \text{Tr}(ZXY)$$

$$\text{Tr}(\Omega) = \text{Tr}(U^\dagger\Omega U)$$

Characteristic Polynomial (1.8.6, Pg 32)

$$\det(\Omega - \omega I) = 0$$

Simultaneous Diagonalization (Theorem 13, Pg 43) If two Hermitian operators commute, there exists a basis of common eigenvectors that diagonalizes them both.

Pauli Matrices The Pauli matrices form an orthogonal basis for the complex Hilbert space of 2x2 matrices.

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With properties (Pg 381):

$$\sigma_i^2 = I, \quad i = 1,2,3$$

$$-i\sigma_1\sigma_2\sigma_3 = I$$

$$[\sigma_a, \sigma_b] = \sum_c 2i\epsilon_{abc}\sigma_c$$

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab}I$$

By 14.3.33 (Pg 381):

$$\sigma_x\sigma_y = i\sigma_z$$

They are orthogonal with respect to the trace inner product (14.3.40a, Pg 382):

$$\text{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$$

with cyclic permutations. Any complex matrix can be written as a linear combination of the Pauli matrices (where $\sigma_0 = \boldsymbol{I}$), (14.3.42-43, Pg 383):

$$M = \sum_a m_a\sigma_a$$

$$m_a = \frac{1}{2}\text{Tr}(M\sigma_a)$$

The Levi-Civita symbol is +1 if the values are cyclic permutations of (1,2,3) and -1 if the values are cyclic permutations of (3,2,1). Eigenvalues are ± 1. Define the Pauli vector $\vec{\sigma}$:

$$\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$$

$$\vec{a}\cdot\vec{\sigma} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

$$(\hat{a}\cdot\vec{\sigma})^{2n} = I$$

$$(\hat{a}\cdot\vec{\sigma})^{2n+1} = \hat{a}\cdot\vec{\sigma}$$

$$e^{i|a|(\hat{a}\cdot\vec{\sigma})} = I\cos|a| + i(\hat{a}\cdot\vec{\sigma})\sin|a|$$

$$\exp\left(\frac{-iHt}{\hbar}\right) = \exp\left(\frac{-i(g\hat{g}\cdot\vec{\sigma})t}{\hbar}\right)$$

$$= \cos\left(\frac{gt}{\hbar}\right)I - i\sin\left(\frac{gt}{\hbar}\right)(\hat{g}\cdot\vec{\sigma})$$

Rotation Matrix (12.2.1, Pg 306)

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates a vector counterclockwise.

Exponential Operator (1.9.3, Pg 54)

$$e^\Omega = \sum_{n=0}^{\infty}\frac{\Omega^n}{n!}$$

$$e^{a\Omega}e^{b\Theta} = e^{a\Omega+b\Theta} \iff [\Omega,\Theta] = 0, \text{Pg } 56$$

Exponential Chain Rule (1.9.8, Pg 56)

$$\frac{d}{d\lambda}\left(e^{\lambda\Omega}e^{\lambda\Theta}\right) = \Omega e^{\lambda\Omega}e^{\lambda\Theta} + e^{\lambda\Omega}e^{\lambda\Theta}\Theta$$

Delta function properties (1.10.23, 1.10.26, Pg 62)

$$\frac{d^n \delta(x-x')}{dx^n} = \delta(x-x')\frac{d^n}{dx'^n}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x'-x)} = \delta(x'-x)$$

More identities from Wolfram Mathworld:

$$\delta'(-x) = -\delta'(x)$$
$$x\delta'(x) = -\delta(x)$$
$$x^n \delta^{(n)}(x) = (-1)^n n!\delta(x)$$
$$\delta(ax) = \frac{1}{|a|}\delta(x)$$
$$\delta(x^2-a^2) = \frac{1}{2|a|}[\delta(x+a)+\delta(x-a)]$$

Fourier Transform and Inverse (1.10.24-25, Pg 62-63)

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx \quad \text{forward}$$
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} F(k)dk \quad \text{reverse}$$

Position and Momentum Space

$$K = -iD \quad \text{is Hermitian, Pg 65}$$
$$\text{Pg 66: } \lim_{x\to\infty} e^{ikx}e^{-ik'x} =$$
$$\lim_{L,\Delta\to\infty}\frac{1}{\Delta}\int_L^{\lambda+\Delta} e^{i(k-k')x}dx = 0, k\neq k'$$
$$\frac{1}{\sqrt{2\pi}}e^{ikx} \quad \text{Momentum eigenfunctions}$$

With $\hbar$ (Pg 137):

$$\psi_p(x) = \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$
$$\psi_x(p) = \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}$$
$$\langle k|K|k'\rangle = k'\delta(k-k') \quad \text{1.10.36, Pg 68}$$
$$\langle x'|X|x\rangle = x\delta(x'-x) \quad \text{1.10.38, Pg 68}$$
$$\langle k|X|k'\rangle = i\delta'(k-k') \quad \text{Pg 69}$$
$$[X,K] = iI, \quad \text{1.10.41}$$

Action (2.1.3, pg 76)

$$S[x(t)] = \int_{t_i}^{t_f}\mathcal{L}(x,\dot{x})dt$$

First order multivariable Taylor expansion (2.1.4, pg 77)

$$f(\boldsymbol{x}^0+\boldsymbol{\eta}) = f(\boldsymbol{x}^0) + \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\boldsymbol{x}^0}\eta_i + \ldots$$

Euler-Lagrange Equation (2.1.9, Pg 78)

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = 0, \quad \forall t_i\leq t\leq t_f$$

Canonical momentum and force (2.1.12-13, pg 80)

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$
$$F_i = \frac{\partial\mathcal{L}}{\partial q_i}$$

Cyclic Coordinate (2.1.14, pg 81) is a coordinate such that the Lagrangian depends on the velocity (time derivative of coordinate) but not the coordinate itself. Then p_i is conserved:

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) = \frac{\partial\mathcal{L}}{\partial q_i} = 0$$

By 2.7.1, Pg 91, for a cyclic coordinate q_i missing in the Hamiltonian:

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} = 0$$

Polar Equations of Motion (2.1.18-19, pg 81)

$$m\ddot{\rho} = -\frac{\partial V}{\partial\rho} + m\rho(\dot{\phi})^2$$
$$m\ddot{\phi} = -\frac{1}{\rho^2}\frac{\partial V}{\partial\phi} - \frac{2m\dot{\rho}\dot{\phi}}{\rho}$$

Lorentz Force CGS (2.2.1, Pg 83)

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)$$

Electromagnetic Lagrangian (2.2.2, Pg 83)

$$\mathcal{L}_{EM} = \frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} - q\phi + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}$$
$$\boldsymbol{p} = m\boldsymbol{v} + \frac{q\boldsymbol{A}}{c}, \quad \text{2.2.7, Pg 84}$$

EM Potentials and Fields (2.2.3-4, Pg 83)

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$$
$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

Two-body CM coordinate (2.3.2-4, Pg 85)

$$\boldsymbol{r}_{CM} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2}$$
$$\boldsymbol{r}_1 = \boldsymbol{r}_{CM} + \frac{m_2\boldsymbol{r}}{m_1+m_2}$$
$$\boldsymbol{r}_2 = \boldsymbol{r}_{CM} - \frac{m_1\boldsymbol{r}}{m_1+m_2}$$

Hamiltonian Formalism (2.5.2, Pg 87)

$$\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

Legendre Transform (2.5.4, Pg 87)

$$\text{Given } u(x) = \frac{df}{dx}$$
$$\text{define } g(u) = x(u)u - f(x(u))$$
$$\text{then } \frac{dg}{du} = x(u)$$

Multidimensional Legendre Transform (2.5.6, Pg 87)

$$\text{Given } f = f(x_1, x_2, \ldots, x_n)$$

we can eliminate a subset of variables in favor of partial derivatives $u_i = \frac{\partial f}{\partial x_i}$ by the transformation:

$$g(u_1,\ldots,u_j,x_{j+1},\ldots,x_n) = \sum_{i=1}^j u_i x_i - f(x_1,\ldots,x_n)$$

so that:

$$\frac{\partial g}{\partial u_i} = x_i$$

Hamiltonian (2.5.14, Pg 89)

$$\mathcal{H} = \sum_i p_i\dot{x}_i - \mathcal{L}$$

Hamilton's Canonical equations (2.5.12, Pg 88)

$$\frac{\partial\mathcal{H}}{\partial p_i} = \dot{q}_i$$
$$-\frac{\partial\mathcal{H}}{\partial q_i} = \dot{p}_i$$

for a total of 2n first-order equations for a system with 2n degrees of freedom.

Electromagnetic Hamiltonian (2.6.2, Pg 91)

$$\mathcal{H}_{EM} = \frac{|(\boldsymbol{p}-q\boldsymbol{A}/c)|^2}{2m} + q\phi$$

Poisson Bracket (2.7.3, Pg 92)

$$\{\omega,\lambda\} \equiv \sum_i\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right)$$

so that:

$$\frac{d\omega}{dt} = \{\omega,\mathcal{H}\}$$

and if the PB vanishes, then ω is conserved. Further identities (Pg 92):

$$\{\omega,\lambda\} = -\{\lambda,\omega\}$$
$$\{\omega,\lambda+\sigma\} = \{\omega,\lambda\} + \{\omega,\sigma\}$$
$$\{\omega,\lambda\sigma\} = \{\omega,\lambda\}\sigma + \lambda\{\omega,\sigma\}$$

Poisson brackets are invariant under canonical transformations (2.7.19, Pg 96).

Poisson Bracket coordinate relations (2.7.4a-4b, Pg 92)

$$\{q_i, q_j\} = \{p_i, p_j\} = 0$$
$$\{q_i, p_j\} = \delta_{ij}$$

These are sufficient conditions for a set of coordinates to be canonical (Pg 95).

Point Transformation (2.7.11, Pg 93)

$$q_i \to \bar{q}_i(q_1, \ldots, q_n)$$
$$\bar{p}_i = \sum_j \frac{\partial q_j}{\partial \bar{q}_i} p_j$$

Regular transformations (Pg 97) preserve the range of the variables.

Symmetries (2.8.3, Pg 99) If $\mathcal{H}$ is invariant under the following infinitesimal canonical transformation:

$$\bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i}$$
$$\bar{p}_i = p_i - \epsilon \frac{\partial g}{\partial q_i}$$

for $g(p, q)$ being any dynamical variable, then g is conserved. Call g the generator of the transformation. Also, by Pg 99, if $\mathcal{H}$ is invariant under the regular, canonical (not necessarily infinitesimal) transformation $(q, p) \to (\bar{q}, \bar{p})$, and if $(q(t), p(t))$ is a solution to the equations of motion, then $(\bar{q}(t), \bar{p}(t))$ also is a solution.

Relation between action S and E (Pg 104)

$$\frac{\partial S_{cl}(x_f, t_f; x_i, t_i)}{\partial t_f} = -\mathcal{H}(t_f)$$

Plane Wave (3.1.1-2, Pg 108)

$$\psi(x,t) = A \exp\left[i \left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t \right) \right]$$
$$\psi(\boldsymbol{r}, t) = A e^{i\boldsymbol{k}\cdot\boldsymbol{r} - \omega t}$$

Postulates of QM (Pg 115-116) I. The state of a system is represented by a vector $|\psi(t)\rangle$ in Hilbert space. II. The Hermitian operators corresponding to position and momentum have the following matrix elements:

$$\langle x|X|x'\rangle = x\delta(x - x')$$
$$\langle x|P|x'\rangle = -i\hbar\delta'(x - x')$$

III. Measurement of Ω yields one of its eigenvalues ω with probability $P(\omega) \propto |\langle\omega|\psi\rangle|^2$. The final state is $|\omega\rangle$. IV. The state vector evolves with Schrodinger's equation (4.3.1, Pg 143):

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Normalized Probability (4.2.1, Pg 118)

$$P(\omega_i) = \frac{|\langle\omega_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

Expectation and Uncertainty (4.2.6-7, Pg 127-128)

$$\langle\Omega\rangle = \langle\psi|\Omega|\psi\rangle$$
$$\Delta\Omega = \sqrt{\langle(\Omega - \langle\Omega\rangle)^2\rangle}$$

Density Matrix (4.2.20-22, Pg 133-134) For general mixed ensembles with a collection of systems in different states.

$$\rho = \sum_i p_i |i\rangle\langle i|$$
$$\langle\bar{\Omega}\rangle = \sum_i p_i \langle i|\Omega|i\rangle$$
$$\mathrm{Tr}(\Omega\rho) = \langle\bar{\Omega}\rangle$$

More identities (4.2.23, Pg 134):

$$\rho^\dagger = \rho$$
$$\mathrm{Tr}(\rho) = 1$$
$$tr(\rho^2) \le 1, \quad \text{equality for pure ensemble}$$

For special ensembles:

$$\rho^2 = \rho \quad \text{pure ensemble}$$
$$\rho = \frac{1}{k} I \quad \text{uniformly distributed ensemble}$$

Normalized Gaussian state (Pg 135) Note that the probability is the magnitude squared.

$$\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-(x-a)^2/2\Delta^2}$$
$$\langle X\rangle = a$$
$$\Delta X = \frac{\Delta}{\sqrt{2}}$$

Electromagnetic Hamiltonian (4.3.6-7, Pg 144)

$$\text{Classical:} \quad \frac{|\boldsymbol{p} - (q/c)\boldsymbol{A}(\boldsymbol{r}, t)|^2}{2m} + q\phi(\boldsymbol{r}, t)$$

Quantum (symmetrized):

$$H = \frac{1}{2m}\left(\boldsymbol{P}\cdot\boldsymbol{P} - \frac{q}{c}\boldsymbol{P}\cdot\boldsymbol{A} - \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{P} + \frac{q^2}{c^2}\boldsymbol{A}\cdot\boldsymbol{A} \right) + q\phi$$

Time evolution propagator (4.3.13-14, Pg 146)

$$U(t) = \sum_E |E\rangle\langle E| e^{-iEt/\hbar}$$
$$U(t) = e^{-iHt/\hbar}$$

so that $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. This operator is unitary, and hence preserves norms. More propagator properties (4.3.16, Pg 149):

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$$
$$U^\dagger(t_2, t_1) = U^{-1}(t_2, t_1) = U(t_1, t_2)$$

Free particle time evolution propagator, position basis (5.1.10, Pg 153):

$$\langle x|U(t)|x'\rangle = \left(\frac{m}{2\pi\hbar i t} \right)^{1/2} e^{im(x-x')^2/2\hbar t}$$

so that the time evolution can be written as the integral (5.1.11, Pg 153):

$$\psi(x,t) = \int \langle x|U(t)|x'\rangle \psi(x', 0) dx'$$

Choosing a basis (Pg 149) In coordinate space (X basis):

$$X \to x \quad P \to -i\hbar \frac{d}{dx}$$

and momentum space (P basis):

$$X \to i\hbar \frac{d}{dp}, \quad P \to p$$

If $V(X)$ is a complicated function of X, use the X basis.

Gaussian Wave Packet (5.1.14, Pg 154)

$$\psi(x', 0) = e^{ip_0 x'/\hbar} \frac{e^{-x'^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$$

For time evolution results (highly complicated) see Pg 154, 5.1.15-16. General results: $\langle X\rangle = \frac{\langle P\rangle t}{m}$, width of packet grows asymptotically like t.

Particle in a box eigenfunctions (5.2.15-16, Pg 159)

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, & \text{n even} \\ \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}, & \text{n odd} \end{cases}$$

Energy levels (5.2.17c, Pg 159):

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

Probability Current Density (5.3.8, Pg 166)

$$\boldsymbol{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^* \right)$$

so that (5.3.7, Pg 166):

$$\frac{\partial P}{\partial t} = -\nabla \cdot \boldsymbol{j}$$

Time Derivative of Expectation Value (6.1, Pg 179)

$$\frac{d}{dt}\langle \Omega \rangle = \langle \dot{\psi}|\Omega|\psi\rangle + \langle \psi|\Omega|\dot{\psi}\rangle + \langle \psi|\dot{\Omega}|\psi\rangle$$

Ehrenfest's Theorem (6.2, Pg 180) for operator without explicit time dependence.

$$\frac{d}{dt}\langle \Omega \rangle = -\frac{i}{\hbar}\langle [\Omega, H]\rangle$$

Hermite Polynomials (7.3.21, Pg 195)

$$H_0(y) = 1$$
$$H_1(y) = 2y$$
$$H_2(y) = -2(1 - 2y^2)$$
$$H_3(y) = -12(y - \frac{2}{3}y^3)$$
$$H_4(y) = 12(1 - 4y^2 + \frac{4}{3}y^4)$$

with recursion relations (7.3.24-25, Pg 195):

$$H_n'(y) = 2nH_{n-1}$$
$$H_{n+1}(y) = 2yH_n - 2nH_{n-1}$$

and orthogonality condition (7.3.26, Pg 195):

$$\int_{-\infty}^{\infty} H_n(y)H_{n'}(y)e^{-y^2}dy = \sqrt{\pi}2^n n!\delta_{nn'}$$

1D Harmonic oscillator solution (7.3.22, Pg 195)

$$\left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} e^{-m\omega x^2/2\hbar} H_n\left[\sqrt{\frac{m\omega}{\hbar}}x\right]$$

Ground state (7.3.40, Pg 200):

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

Raising and Lowering Operators (7.4.3-4, Pg 203):

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2m\omega\hbar}}P$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2m\omega\hbar}}P$$

with commutation relation (7.4.5, Pg 203):

$$[a, a^\dagger] = 1$$

We can re-write the 1D Hamiltonian (7.4.6, Pg 204):

$$H = (a^\dagger a + 1/2)\hbar\omega$$

The operation of these are (7.4.21-22, Pg 207):

$$a|n\rangle = \sqrt{n}|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Inverting the operators (7.4.28-29, Pg 207):

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$
$$P = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$$

Commutation relations for N degrees of freedom (7.4.39, Pg 211)

$$[X_i, P_j] = i\hbar\delta_{ij} = i\hbar\{x_i, p_j\}_{PB}$$
$$[X_i, X_j] = 0 = i\hbar\{x_i, x_j\}_{PB}$$
$$[P_i, P_j] = 0 = i\hbar\{p_i, p_j\}_{PB}$$

Use these relations to promote parameters to operators and quantize the system.

Path Integral formulation of propagator (8.1.1, Pg 223)

$$U(x, t; x', t') = A \sum_{\text{all paths}} e^{iS[x(t)]/\hbar}$$

where A is a normalization factor. Note that the sum is dominated by the classical action because the action is stationary around the classical path, allowing the contributions to add constructively. Formally, (8.4.1, Pg 226):

$$\int_{x_0}^{x_N} e^{iS[x(t)]/\hbar}\mathcal{D}[x(t)]$$

where the sum over all paths is performed as (8.4.8, Pg 229):

$$\int \mathcal{D}[x(t)] =$$
$$\lim_{\epsilon\to 0, N\to\infty} \frac{1}{B}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1}{B}\cdots\frac{dx_{N-1}}{B}$$
$$B = \left(\frac{2\pi\hbar\epsilon i}{m}\right)^{1/2}$$

General Uncertainty Relation (9.2.12, Pg 239)

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq$$
$$\frac{1}{4}\langle\psi|\{\hat{\Omega}, \hat{\Lambda}\}|\psi\rangle^2 + \frac{1}{4}\langle\psi|[\hat{\Omega}, \hat{\Lambda}]|\psi\rangle^2$$

Two particle QM (Pg 247-248)

$$[X_i, P_j] = i\hbar\delta_{ij} \quad 10.1.1a$$
$$[X_i, X_j] = 0 \quad 10.1.1b$$
$$[P_i, P_j] = 0 \quad 10.1.1c$$
$$\langle x_1' x_2'|x_1 x_2\rangle = \delta(x_1' - x_1)\delta(x_2' - x_2)$$
$$P(x_1, x_2) = |\langle x_1 x_2|\psi\rangle|^2 \quad 10.1.5$$
$$\int P(x_1, x_2)dx_1 dx_2 = 1 \quad 10.1.6$$

Direct Product (10.1.10, Pg 249) is the product of vectors from two different spaces. It is linear:

$$(\alpha|x_1\rangle + \alpha'|x_1'\rangle) \otimes (\beta|x_2\rangle)$$
$$= \alpha\beta|x_1\rangle \otimes |x_2\rangle + \alpha'\beta|x_1'\rangle \otimes |x_2\rangle$$

The inner product in the direct product space is (10.1.11, Pg 250):

$$(\langle x_1'| \otimes \langle x_2'|)(|x_1\rangle \otimes |x_2\rangle)$$
$$= \langle x_1'|x_1\rangle\langle x_2'|x_2\rangle = \delta(x_1' - x_1)\delta(x_2' - x_2)$$

The coordinate-space representation is (10.1.22a, Pg 253):

$$\psi(x_1, x_2) = \sum_{\omega_1}\sum_{\omega_2} C_{\omega_1,\omega_2}\omega_1(x_1)\omega_2(x_2)$$

which corresponds to (10.1.22b, Pg 253):

$$|\psi\rangle = \sum_{\omega_1}\sum_{\omega_2} C_{\omega_1,\omega_2}|\omega_1\rangle \otimes |\omega_2\rangle$$

Direct Product of Two Operators (10.1.14, Pg 250)

$$(\Gamma_1^{(1)} \otimes \Lambda_2^{(2)})|\omega_1\rangle \otimes |\omega_2\rangle$$
$$= |\Gamma_1^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle$$

Define the operator $\Omega_i^{(j)}$ as operating on the i particle in the vector space j. Note that this operator acts as the identity on all spaces not j. The direct product has properties (10.1.17a-17d, Pg 251):

$$[\Omega_1^{(1)} \otimes I^{(2)}, I^{(1)} \otimes \Lambda_2^{(2)}] = 0$$
$$(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\Theta_1^{(1)} \otimes \Lambda_2^{(2)})$$
$$= (\Omega\Theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}$$
$$\text{If } [\Omega_1^{(1)}, \Lambda_1^{(1)}] = \Gamma_1^{(1)},$$
$$\text{Then } [\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}] = \Gamma_1^{(1)} \otimes I^{(2)}$$
$$(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)})^2$$
$$= (\Omega_1^2)^{(1)} \otimes I^{(2)} + I^{(1)} \otimes (\Omega_2^2)^{(2)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)}$$

and commutation relations (10.1.18, Pg 252):

$$[X_i^{(1)\otimes(2)}, P_j^{(1)\otimes(2)}] = i\hbar\delta_{ij}I^{(1)\otimes(2)}$$
$$[X_i^{(1)\otimes(2)}, X_j^{(1)\otimes(2)}] = 0$$
$$[P_i^{(1)\otimes(2)}, P_j^{(1)\otimes(2)}] = 0$$

Time evolution of multi-particle system For separable Hamiltonian $\mathcal{H} = \mathcal{H}_1(x_1, p_1) + \mathcal{H}_2(x_2, p_2)$ (10.1.29, Pg 255):

$$|\psi(t)\rangle = |E_1\rangle e^{-iE_1t/\hbar} \otimes |E_2\rangle e^{-iE_2t/\hbar}$$

Two Interacting Particles (Pg 256-257) For two particles with potential that only depends on the separation $V(x_1 - x_2)$, move into CM coordinates with separation operators $x = x_1 - x_2$, $p = \mu\dot{x}$, where $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass. Then the Hamiltonian is (10.1.38, Pg 257):

$$\mathcal{H} = \frac{p_{CM}^2}{2M} + \frac{p^2}{2\mu} + V(x)$$

with quantization conditions (commutator relations) from 10.1.39a-b, Pg 257:

$$[X_{CM}, P_{CM}] = i\hbar$$
$$[X, P] = i\hbar$$

with all other commutators zero. In the position basis, the energy eigenfunctions become (10.1.41, Pg 257):

$$\psi_E(x_{CM}, x) = \frac{e^{ip_{CM}\cdot x_{CM}/\hbar}}{\sqrt{2\pi\hbar}}\psi_{E_{rel}}(x)$$
$$E = \frac{p_{CM}^2}{2M} + E_{rel}$$

We can move into the CM frame (assuming CM drifts like a free particle) and set $p_{CM} = 0$.

Bosonic and Fermionic Hilbert spaces (Pg 265-268) The normalized symmetric eigenvector (discrete variable) is (10.3.10a):

$$|\omega_1\omega_2, S\rangle = \begin{cases} \frac{(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle)}{\sqrt{2}}, \omega_1 \neq \omega_2 \\ |\omega\omega\rangle, \quad \omega_1 = \omega_2 = \omega \end{cases}$$

with probability function (10.3.11, Pg 266):

$$P_S(\omega_1, \omega_2) = |\langle\omega_1\omega_2, S|\psi_S\rangle|^2$$

and normalization condition (10.3.12a-b, Pg 266):

$$\sum_{dist} P_S(\omega_1, \omega_2) = 1$$
$$\sum_{\omega_2=\omega_{min}}^{\omega_{max}} \sum_{\omega_1=\omega_{min}}^{\omega_2} P_S(\omega_1, \omega_2) = 1$$

where the summation is over all distinct states.

The wavefunction in position space has a scale factor (10.3.16, Pg 267), and similarly for the asymmetric case:

$$\psi_S(x_1, x_2) = \frac{1}{\sqrt{(2!)}}\langle x_1x_2, S|\psi_s\rangle$$

so that the normalization can be performed over all 2D space (10.3.17, Pg 267):

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_S(x_1, x_2)|^2 dx_1dx_2$$

Note that the probability must be written with the scale factor if not over the entire space (10.3.18), Pg 267):

$$P_S(x_1, x_2) = 2!|\psi_S(x_1, x_2)|^2$$

Translational invariance (11.2.40, Pg 292)

$$H(X, P) = T^\dagger(\epsilon)HT(\epsilon) = H(X + \epsilon I, P)$$

Generators and Operators (11.2.13, Pg 283)

$$T(\epsilon) = I - \frac{i\epsilon}{\hbar}G$$

G is the generator and is Hermitian.

Unitary transformation of operator (Pg 286) For any $\Omega(X, P)$ that can be expanded in a power series, for any unitary operator U,

$$U^\dagger\Omega(X, P)U = \Omega(U^\dagger XU, U^\dagger PU)$$

Finite translation operator (11.2.29, Pg 290):

$$T(a) = e^{-iaP/\hbar}$$

Parity transformation (11.4.2-3, Pg 297)

$$\Pi|x\rangle = |-x\rangle$$
$$\Pi|p\rangle = |-p\rangle$$

and its operation on operators (11.4.7, Pg 298):

$$\Pi^\dagger X\Pi = -X$$
$$\Pi^\dagger P\Pi = -P$$

Parity invariance is defined as (11.4.8, Pg 298):

$$\Pi^\dagger H\Pi = H(-X, -P) = H(X, P)$$

In spherical coordinates (need to prove this):

$$r \to r$$
$$\theta \to \pi - \theta$$
$$\phi \to \pi + \phi$$

The net effect of the parity transformation is to achieve:

$$Y_l^m(\pi - \theta, \pi + \phi) = (-1)^l Y_l^m(\theta, \phi)$$

so the parity is $(-1)^l$. This can be proved by examining the Y_l^l function (12.5.32, Pg 335):

$$Y_l^l \propto e^{il\phi}\sin^l\theta$$

Then show that $[L_-, \Pi] = 0$ so that the parity doesn't change under lowering.

Translation in arbitrary direction (12.1.5, Pg 306):

$$T(\vec{a}) = e^{-i\vec{a}\cdot\vec{P}/\hbar}$$

Angular momentum operators (12.2.11, Pg 308):

$$L_z = XP_y - YP_x$$

and in polar coordinates (12.2.19, Pg 309):

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

It has normalized eigenfunctions (12.3.9-10, Pg 315):

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$
$$\int_0^{2\pi} \Phi_m^*\Phi_{m'}d\phi = \delta_{mm'}$$

In the other directions (12.4.1ab, Pg 318):

$$L_x = YP_z - ZP_y$$
$$L_y = ZP_x - XP_z$$

Angular momentum commutators (12.2.16-17, Pg 309)

$$[X, L_z] = -i\hbar Y$$
$$[Y, L_z] = i\hbar X$$
$$[P_x, L_z] = -i\hbar P_y$$
$$[P_y, L_z] = i\hbar P_x$$

and (12.4.4abc, Pg 318):

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

or in summary (12.4.5-6, Pg 319):

$$\vec{L} \times \vec{L} = i\hbar\vec{L}$$
$$[L_i, L_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}L_k$$

Cross Product and Levi-Civita symbol (12.4.8-9, Pg 319):

$$\vec{c} = \vec{a} \times \vec{b}$$
$$c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_j b_k$$

Rotation generator (12.4.12, Pg 320):

$$U[R(\Theta)] = e^{-i\Theta\cdot\vec{L}/\hbar}$$

Angular momentum raising and lowering operators (12.5.3, Pg 322):

$$L_\pm = L_x \pm iL_y$$

with commutation relations (12.5.4-5, Pg 322):

$$[L_z, L_\pm] = \pm\hbar L_\pm$$
$$[L^2, L_\pm] = 0$$

Inverting,

$$J_x = \frac{J_+ + J_-}{2}$$
$$J_y = \frac{J_+ - J_-}{2i}$$

These operators give the effect (12.5.20, Pg 327):

$$J_\pm|jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

In the coordinate basis (12.5.27, Pg 334),

$$L_\pm \to \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Angular momentum eigenvalue equations (12.5.17ab, Pg 326):

$$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle, j = 0, 1/2, 1\ldots$$
$$J_z|jm\rangle = m\hbar|jm\rangle, \quad m = j, j-1, \ldots, -j$$

For orbital angular momentum (12.5.18ab, Pg 326):

$$L^2|lm\rangle = l(l+1)\hbar^2|lm\rangle, l = 0, 1, 2, \ldots$$
$$L_z|lm\rangle = m\hbar|lm\rangle, m = l, l-1, \ldots, -l$$

Spherical harmonics Orthonormality condition (Pg 335):

$$\int [Y_l^m(\theta,\phi)]^* Y_{l'}^{m'}(\theta,\phi) d\Omega = \delta_{ll'}\delta_{mm'}$$
$$d\Omega = d(\cos\theta)d\phi$$

Any wavefunction can be expanded in spherical harmonics (12.5.37ab, Pg 336):

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_l^m(r) Y_l^m(\theta,\phi)$$
$$C_l^m(r) = \int [Y_l^m(\theta,\phi)]^* \psi(r,\theta,\phi) d\Omega$$

See page 337 for a list of spherical harmonics. The angular momentum operators have the effect:

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$
$$L_z Y_l^m = \hbar m Y_l^m$$

The physical meaning of the coefficients are the probabilities that a wavefunction has particular angular momentum eigenvalues ($L^2 = l(l+1)\hbar, L_z = m\hbar$):

$$P_{l,m} = \int_0^\infty |C_l^m(r)|^2 r^2 dr$$

Negative m values correspond to (12.5.40, Pg 337):

$$Y_l^{-m} = (-1)^m [Y_l^m]^*$$

Rotationally invariant problems in spherical coordinates The Laplacian is (12.6.1, Pg 339):

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

which can be written using the spherical form of the total angular momentum L^2:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$$

We seek simultaneous eigenfunctions of H, L^2 and L_z, which have the form (12.6.2, Pg 339):

$$\psi_{Elm} = R_{Elm}(r) Y_l^m(\theta,\phi)$$

Make the change of variable (12.6.4, Pg 340):

$$R_{El} = \frac{U_{El}}{r}$$

and at $r \to 0$, the asymptotic solution is (12.6.14, Pg 343):

$$U_l \sim r^{l+1}, r^{-l}$$

Reject the irregular solutions r^{-l} since they do not meet the boundary condition (12.6.12, Pg 342):

$$U_{El} \to 0, r \to 0$$

Auxiliary radial function for spherically symmetric potentials (12.6.4-5, Pg 340)

$$R_{El} = \frac{U_{El}}{r}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]\right\} U_{El} = 0$$

We also require at the boundaries (12.6.8ab, Pg 341, 12.6.12, Pg 242):

$$\lim_{r\to\infty} U_{El} = \begin{cases} 0, & E < 0 \\ e^{ikr}, & E > 0 \end{cases}$$
$$\lim_{r\to 0} U_{El} = 0$$

Free particle in 3D (12.6.36, Pg 349)

$$\psi_{Elm}(r,\theta,\phi) = j_l(kr) Y_l^m(\theta,\phi)$$

where $E = \frac{\hbar^2 k^2}{2\mu}$ and j_l is the spherical Bessel function of order l.

Free particle, Spherically symmetric differential equation (12.6.20, Pg 346) Let $\rho = kr, k^2 = \frac{2\mu E}{\hbar^2}, R = \frac{U_{El}}{r}$:

$$\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2}\right] U_l = U_l$$

3D Isotropic Harmonic Oscillator Energies (12.6.50, Pg 352):

$$E = (2k + l + 3/2)\hbar\omega, k = 0, 1, 2, \ldots$$
$$n \equiv 2k + l$$
$$l = n, n-2, \ldots, 1 \text{ or } 0$$
$$m = -l, -l+1, \ldots, l$$

Hydrogen Atom Principal quantum number (13.1.15, Pg 355):

$$n = k + l + 1 = 1, 2, 3, \ldots$$

where k is the index that the summation terminates at. Energies (13.1.14, Pg 355):

$$E = -\frac{me^4}{2\hbar^2(k+l+1)^2}$$
$$k = 0, 1, 2, \ldots, n-1$$
$$l = 0, 1, 2, \ldots, n-1$$

Degeneracy for each n (excluding spin), (13.1.18, Pg 355):

$$\sum_{l=0}^{n-1}(2l+1) = n^2$$

Bohr radius (13.1.24, Pg 357):

$$a_0 = \frac{\hbar^2}{me^2}$$

Radial equation (13.1.25, Pg 357):

$$R_{nl} \sim e^{-r/na_0} \left(\frac{r}{na_0}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na_0}\right)$$

where L is the associated Laguerre polynomial. It has $n-l-1$ zeros.

Virial theorem (13.1.33, Pg 358) For the Coulomb force:

$$\langle T \rangle = -\frac{1}{2}\langle V \rangle$$

More generally (Exercise 13.1.5, Pg 359):

$$2\langle T \rangle = \left\langle r_i \frac{\partial V}{\partial r_i}\right\rangle$$

Spin commutation relations (14.3.4, Pg 375)

$$[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k$$

This applies for L and S as well.

Spin-half spinor (14.3.8a, Pg 375)

$$\langle x, y, z, s_z|\psi\rangle = \begin{bmatrix} \psi_+(x,y,z) \\ \psi_-(x,y,z) \end{bmatrix}$$

with normalization constraint (14.3.14, Pg 377):

$$\int (|\psi_+|^2 + |\psi_-|^2) dxdydz = 1$$

Spin along arbitrary axis, electron (14.3.27, Pg 380)

$$\hat{n}\cdot\boldsymbol{S} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

The eigenstates are (14.3.28ab, Pg 380):

$$|\hat{n}, +\rangle = \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

$$|\hat{n}, -\rangle = \begin{bmatrix} -\sin\frac{\theta}{2} e^{-i\phi/2} \\ \cos\frac{\theta}{2} e^{i\phi/2} \end{bmatrix}$$

Unitary rotation operator (14.3.44, Pg 383)

$$U[R(\boldsymbol{\theta})] = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{S}/\hbar}$$

$$U[R(\boldsymbol{\theta})] = \cos\frac{\theta}{2}\boldsymbol{I} - i(\hat{\theta}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}$$

Hamiltonian in magnetic field (14.4.11, Pg 387)

$$\frac{|\boldsymbol{P}|^2}{2m} - \frac{q}{2mc}(\boldsymbol{P}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{P}) + \frac{q^2|\boldsymbol{A}|^2}{2mc^2}$$

Interaction Hamiltonian (14.4.14, Pg 388):

$$H_{int} = -\boldsymbol{\mu}\cdot\boldsymbol{B}$$

and for the electron (14.4.19, Pg 389),

$$H_{int} = \frac{ge\hbar}{4mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

Gyromagnetic ratio (14.4.15, Pg 388):

$$\boldsymbol{\mu} = \frac{q}{2mc}\boldsymbol{L}$$

$$\gamma \equiv \frac{q}{2mc}$$

The precession frequency is $\boldsymbol{\omega}_0 = -\gamma\boldsymbol{B}$ so that (14.4.24, Pg 391):

$$\boldsymbol{\theta}(t) = -\gamma\boldsymbol{B}t$$

Addition of angular momenta (15.2.4, Pg 409) Let $j_1 \geq j_2$. Then the eigenkets of the total $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$ operator are:

$$|jm\rangle, \quad j_1 - j_2 \leq j \leq j_1 + j_2, -j \leq m \leq j$$

Obtaining the Clebsch-Gordon coefficients (Pg 408-412) 1. Consider the maximum m state of the maximum value of $j = j_1 + j_2$. Hence $m = j_1 + j_2$. There is only one state that achieves this. 2. Apply the total lowering operator $S_- = S_{1-} + S_{2-}$ to obtain the state with $m = j_1 + j_2 - 1$. Expand the effect of S_- in terms of the individual constituent eigenstates. 3. Repeat until you get to $m = j_1 - j_2$. 4. Move to the next value of $j = j_1 + j_2 - 1$. Declare that it should be a superposition of $|j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle$ and $|j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle$, and should be orthogonal to the state $|j_1 + j_2, j_1 + j_2 - 1\rangle$ obtained earlier. Solve for the real coefficients, then proceed with the total lowering operator.

Clebsch-Gordon coefficients properties (15.2.9-11, Pg 412-413) They are the overlap values:

$$\langle j_1 m_1, j_2 m_2 | jm\rangle$$

and satisfy:

$$\langle j_1 m_1, j_2 m_2 | jm\rangle \neq 0,$$
$$\text{only if } j_1 - j_2 \leq j \leq j_1 + j_2$$
$$\text{and } m_1 + m_2 = m$$

with the convention that they are real,

$$\langle j_1 j_1, j_2 (j - j_1) | jj\rangle > 0$$

and the effect of flipping the sign of m is:

$$(-1)^{j_1 + j_2 - j}\langle j_1(-m_1), j_2(-m_2)|j(-m)\rangle$$

Spectroscopic notation (Pg 415) Denote the angular momentum by $L = SPDF$ for $l = 0, 1, 2, 3$. Then write:

$$^{2S+1}L_J$$

Scalar, vector and tensor operators (Pg 416-417) A scalar operator S remains invariant under rotations:

$$S' = U^\dagger[R] S U[R] = S$$

A vector operator is a collection of three operators $\boldsymbol{V} = (V_x, V_y, V_z)$ which transform as the components of a vector under rotations:

$$V_i' = U^\dagger[R] V_i U[R] = \sum_j R_{ij} V_j$$

A tensor operator of rank 2 is a collection of nine operators T_{ij} which respond as to the basis kets $|i\rangle \otimes |j\rangle$ under rotations.

Variational method for finding ground state (Pg 429-435) 1. Make a guess for the ground state wavefunction with some unknown parameter. 2. Calculate:

$$E[\psi] = \frac{\int \psi^* H \psi dx}{\int \psi^* \psi dx}$$

3. Minimize this function with respect to the unknown parameter. The parameter that minimizes the energy is the best estimate given that particular wavefunction form guess.

WKB approximation (Pg 435-438) To solve:

$$\left[\frac{d^2}{dx^2} + \frac{p^2(x)}{\hbar^2}\right]\psi(x) = 0$$

where $p^2(x) = 2m[E - V(x)]$, make the guess:

$$\psi(x) = e^{i\phi(x)/\hbar}$$

Then expand the phase in orders of $\hbar$:

$$\phi = \phi_0 + \hbar\phi_1 + \hbar^2\phi_2 + \ldots$$

Keeping only the linear terms, and requiring that the first and second order terms proportional to $\hbar^{-1}, \hbar^{-2}$ all vanish. Then the wavefunction approximation is:

$$\psi(x) = \psi(x_0)\sqrt{\frac{p(x_0)}{p(x)}}\exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x} p(x')dx'\right]$$

This approximation holds when the wavefunction wavelength is much smaller than the characteristic potential lengthscale (hence we have to avoid classical turning points when the wavelength is large).

Difference between perturbation schemes To find the new eigenstates and eigenvalues of a perturbed Hamiltonian, use time-independent perturbation theory. To find the transition amplitude between orthogonal eigenstates in a perturbed Hamiltonian, use time-dependent perturbation theory.

Time-Independent Perturbation theory (Pg 451-453) Given a solved Hamiltonian:

$$H^0|n^0\rangle = E_n^0|n^0\rangle$$

The energy eigenvalues for the perturbed Hamiltonian $H = H^0 + H^1$ are:

$$E_n = E_n^0 + E_n^1 + E_n^2 + \dots$$
$$E_n^1 = \langle n^0|H^1|n^0\rangle$$
$$E_n^2 = \sum_{m\neq n} \frac{|\langle n^0|H^1|m^0\rangle|^2}{E_n^0 - E_m^0}$$

(note E_n^2 is the second order energy correction, not a square!) and the eigenstates are:

$$|n\rangle = |n^0\rangle + |n^1\rangle + \dots$$
$$|n^1\rangle = \sum_{m\neq n} \frac{|m^0\rangle\langle m^0|H^1|n^0\rangle}{E_n^0 - E_m^0}$$

The necessary condition for $|n^1\rangle$ to be small compared to $|n^0\rangle$ is (17.1.18, Pg 454):

$$\left|\frac{\langle m^0|H^1|n^0\rangle}{E_n^0 - E_m^0}\right| << 1$$

Selection rules (17.2.12, Pg 458) If $[\Omega, H^1] = 0$, then:

$$\langle\alpha_2\omega_2|H^1|\alpha_1\omega_2\rangle = 0$$

unless $\omega_1 = \omega_2$, the eigenvalues of Ω for the two states are equal.

Parity selection rule (17.2.20, Pg 459) If $\Pi^\dagger\Omega\Pi = -\Omega$ then the matrix element of Ω between two parity eigenstates vanishes unless they have opposite parity.

Dipole selection rule (17.2.21, Pg 459)

$$\langle l_2m_2|Z|l_1m_1\rangle = 0 \text{ unless } \begin{cases} l_2 = l_1 \pm 1 \\ m_2 = m_1 \end{cases}$$

$$\langle l_2m_2|Y|l_1m_1\rangle = 0 \text{ unless } \begin{cases} l_2 = l_1 \pm 1 \\ m_2 = m_1 \pm 1 \end{cases}$$

$$\langle l_2m_2|X|l_1m_1\rangle = 0 \text{ unless } \begin{cases} l_2 = l_1 \pm 1 \\ m_2 = m_1 \pm 1 \end{cases}$$

Degenerate perturbation theory (Pg 464-466) When there exists a degenerate subspace with nonzero operator matrix elements, we have to find the basis that diagonalizes the perturbation H^1 in that subspace. The eigenstates that diagonalize H^1 are stable under the perturbation and hence TIPT can be used to calculate their energy shifts.

Time-dependent perturbation theory, first order (Pg 474-475) Consider a Hamiltonian $H = H^0 + H^1(t)$. Write the wavefunction as a superposition of original eigenstates:

$$|\psi(t)\rangle = \sum_n d_n(t)e^{-iE_n^0t/\hbar}|n^0\rangle$$

If the system is initially in the eigenstate $|i^0\rangle$ at $t = 0$, the first order approximation for the coefficients is (18.2.9, Pg 475):

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle f_0|H^1(t')|i^0\rangle e^{i\omega_{fi}t'}dt'$$

where $\omega_{fi} = \frac{E_f^0 - E_i^0}{\hbar}$. The transition probability is hence:

$$P_{i\to f} = |d_f(t)|^2$$

The condition for this approximation to hold is:

$$|d_f(t)| << 1, \quad f \neq i$$

Adiabatic theorem (Pg 478-479) Let a Hamiltonian change slowly from $H(0)$ to $H(\tau)$ in time τ. If a system starts out in an eigenstate $|n(0)\rangle$ of $H(0)$, if the rate of change is slow enough, it will end up in the corresponding eigenket $|n(\tau)\rangle$ of $H(\tau)$.

Fermi's golden rule (18.2.42, Pg 483) Let a system be subject to a periodic perturbation $H^1(t) = H^1e^{-i\omega t}$ for a long time $-T/2 \leq t \leq T/2$. The average transition rate between eigenstates $|i^0\rangle$ and $|f^0\rangle$ is:

$$R_{i\to f} = \frac{2\pi}{\hbar}|\langle f^0|H^1|i^0\rangle|^2\delta(E_f^0 - E_i^0 - \hbar\omega)$$

where the rate is defined as:

$$R_{i\to f} = \frac{P_{i\to f}}{T}$$

Unitary Propagator (Pg 484-485) Given the Schrodinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

we define the propagator:

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$$

which must satisfy (18.3.5, Pg 485):

$$i\hbar\frac{dU}{dt} = H(t)U$$

Interaction Picture (Pg 485-490) Consider the perturbed Hamiltonian $H_s(t) = H_s^0 + H_s^1(t)$ in the Schrodinger picture (particle described by state vector). The unperturbed unitary propagator satisfies:

$$i\hbar\frac{dU_s^0}{dt} = H_s^0U_s^0$$

Define the state vector in the interaction picture (also applies to eigenstates):

$$|\psi_I(t)\rangle = [U_s^0(t,t_0)]^\dagger|\psi_s(t)\rangle$$

and the transformed Hamiltonian perturbation:

$$H_I^1(t) = (U_s^0)^\dagger H_s^1(t)U_s^0$$

The time evolution of the state in the interaction picture is hence (18.3.10, Pg 486):

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = H_I^1(t)|\psi_I(t)\rangle$$

The propagator in the interaction picture can be calculated to first order (18.3.21, Pg 488):

$$U_I(t,t_0) = I - \frac{i}{\hbar}\int_{t_0}^t H_I^1(t')dt'$$

and the Schrodinger picture propagator can hence be obtained with (18.3.18, Pg 487):

$$U_s(t,t_0) = U_s^(t,t_0)U_I(t,t_0)$$

Heisenberg picture (Pg 490-491) The Heisenberg state vector does not change in time:

$$|\psi_H(t)\rangle = U_s^\dagger(t,t_0)|\psi_s(t)\rangle = |\psi_s(t_0)\rangle$$

Note that the full propagator is used, not just the unperturbed propagator. Operators in the Heisenberg picture are hence transformed as (18.3.29, Pg 490):

$$\Omega_H(t) = U_s^\dagger\Omega_sU_s$$

and have time derivatives (18.3.30, Pg 490):

$$i\hbar\frac{d\Omega_H}{dt} = [\Omega_H, H_H]$$

Electromagnetic gauge transformations (18.4.12-13, Pg 493)

$$A' = A - \nabla\Lambda$$
$$\phi' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

Coulomb Gauge (18.4.14-15, Pg 494)

$$\nabla \cdot \boldsymbol{A} = 0$$
$$\phi = 0$$
$$|A| \to 0, \quad \text{at infinity}$$

which gives solutions (18.4.17, Pg 494):

$$\boldsymbol{A} = \boldsymbol{A}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

and dispersion relation (18.4.18):

$$\omega = kc$$

and electromagnetic field (18.4.20-21, Pg 495):

$$\boldsymbol{E} = -\frac{\omega}{c}\boldsymbol{A}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$
$$\boldsymbol{B} = -(\boldsymbol{k}\times\boldsymbol{A}_0)\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

The Poynting vector is defined (18.4.23a, Pg 495):

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}$$

with time average (18.4.23b, Pg 496):

$$S_{av} = \frac{\omega^2}{8\pi c}|\boldsymbol{A}_0|^2$$

The field energy density is (18.4.24, Pg 496):

$$u = \frac{1}{8\pi}(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)$$

Effect of gauge transformations in Q The action changes by (18.4.29, Pg 496):

$$S_\Lambda = S + \frac{q}{c}[\Lambda(\boldsymbol{r}', t') - \Lambda(\boldsymbol{r}, t)]$$

which by the path integral approach, changes the propagator by (18.4.30, Pg 497):

$$U_\Lambda = U \exp\left\{\frac{iq}{\hbar c}\left[\Lambda(\boldsymbol{r}', t') - \Lambda(\boldsymbol{r}, t)\right]\right\}$$

which just changes the coordinate basis (18.4.32, Pg 497):

$$|\boldsymbol{r}_\Lambda\rangle = e^{iq\Lambda/\hbar c}|\boldsymbol{r}\rangle$$

Electric dipole approximation (18.5.12, Pg 502)

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \approx 1$$

Asymptotic scattering eigenfunction (19.2.10, Pg 527)

$$\psi_{sc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{kr}\sum_{l,m}(-i)^l B_l Y_l^m(\theta,\phi)$$
$$\psi_{\boldsymbol{k}} \xrightarrow[r\to\infty]{} e^{ikz} + \psi_{sc}$$

Call the coefficient of $\frac{e^{ikr}}{r}$ the scattering amplitude $f(\theta,\phi)$. Then the scattering cross section is (19.2.18, Pg 529):

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

so that the probability flow rate as a function of solid angle is (19.2.17, Pg 529):

$$R(d\Omega) = |f|^2\frac{\hbar k}{\mu}d\Omega$$

Probability flux of plane wave (19.3.3, Pg 530) Consider a plane wave $|\boldsymbol{p}_i\rangle \to (2\pi\hbar)^{-3/2}e^{i\boldsymbol{p}_i\cdot\boldsymbol{r}/\hbar}$. Then the probability flux is:

$$j = \frac{\hbar k}{\mu}\frac{1}{(2\pi\hbar)^{3/2}}$$

Time-dependent Born approximation Define the S matrix (Pg 529):

$$S = \lim_{t_f\to\infty, t_i\to-\infty} U(t_f, t_i)$$

so that the transition rate (to first order, using Fermi's golden rule) is (19.3.2, Pg 530):

$$R_{i\to d\Omega} = \frac{2\pi}{\hbar}|\langle\boldsymbol{p}_f|V|p_i\rangle|^2\mu p_i d\Omega$$

where p_i is the incoming momentum of a plane wave and $\boldsymbol{p}_f$ is the momentum so that the particle enters a detector located at (θ,ϕ) with opening angle $d\Omega$. The scattering cross section is (19.3.4, Pg 530):

$$\frac{d\sigma}{d\Omega} = \left|\frac{\mu}{2\pi\hbar^2}\int e^{-i\boldsymbol{q}\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d^3\boldsymbol{r}'\right|^2$$

where $\hbar\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ represents the momentum transferred to the particle, and which has magnitude (19.3.6, Pg 530):

$$|\boldsymbol{q}|^2 = 4k^2\sin^2\frac{\theta}{2}$$

where k is the magnitude of the wavenumber of the incoming or outgoing wave (should be the same since the energy must be the same at infinity). The Born approximation gives the scattering amplitude (19.3.7, Pg 530):

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\boldsymbol{q}\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d^3\boldsymbol{r}'$$

If the potential is spherically symmetric, we can choose z' to be parallel to $\boldsymbol{q}$ so that (19.3.8, Pg 531):

$$f(\theta) = -\frac{2\mu}{\hbar^2}\int\frac{\sin qr'}{q}V(r')r'dr'$$

where the θ dependence is contained in q.

Time-independent Born Approximation Let G^0 be the Green's function for the Schrodinger equation (19.4.1, Pg 534):

$$(\nabla^2 + k^2)\psi_{\boldsymbol{k}} = \frac{2\mu}{\hbar}V\psi_{\boldsymbol{k}}$$

so that (19.4.3, Pg 534):

$$(\nabla^2 + k^2)G^0(\boldsymbol{r},\boldsymbol{r}') = \delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

and the formal solution to the wavefunction will be (19.4.4, Pg 535):

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \psi^0(\boldsymbol{r})$$
$$+\frac{2\mu}{\hbar}\int G^0(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi_{\boldsymbol{k}}(\boldsymbol{r}')d^3\boldsymbol{r}'$$

where the homogeneous solution satisfies (19.4.5, Pg 535):

$$(\nabla^2 + k^2)\psi^0 = 0$$

and can be written (19.4.6, Pg 535):

$$\psi^0 = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

The Green's function can be found (19.4.16, Pg 538):

$$G^0(\boldsymbol{r},\boldsymbol{r}') = -\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Validity of Born approximation (19.4.44, Pg 544) when:

$$\frac{2\mu}{\hbar^2 k}\left|\int e^{ikr'}\sin kr' V(r')dr'\right| << 1$$

Plane wave expansion (19.5.3, Pg 545)

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$$

Asymptotic behavior of spherical Bessel and Neumann functions (19.2.9, Pg 527)

$$j_l(kr) \to \frac{\sin(kr - l\pi/2)}{kr}$$
$$\eta_l(kr) \to -\frac{\cos(kr - l\pi/2)}{kr}$$

in the limit $r \to \infty$.

Partial wave expansion First note the relationship between Legendre polynomials of trigonometric functions and spherical harmonics (19.5.1, Pg 545):

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_l^0$$

then expand the scattering amplitude in Legendre polynomials (19.5.2, Pg 545):

$$f(\theta, k) = \sum_{l=0}^{\infty} (2l+1) a_l(k) P_l(\cos\theta)$$

where a_l is the lth partial wave amplitude, which is a measure of the scattering for components with angular momentum l. Now require that the radial wavefunctions in the presence of a potential reduce to the free-particle wavefunction at $r \to \infty$, up to a phase $\delta_l(k)$ (19.5.9, Pg 546):

$$R_l(r) \to \frac{A_l \sin[kr - l\pi/2 + \delta_l(k)]}{r}$$

The expansion coefficients are (19.5.14, Pg 547):

$$a_l(k) = \frac{e^{2i\delta_l(k)} - 1}{2ik}$$

which hence reduces the scattering problem to finding the phase shifts. Repulsive potentials give negative phase shifts and attractive potentials give negative phase shifts (Pg 550). Define the partial wave S matrix element for angular momentum l (19.5.15, Pg 546):

$$S_l(k) = e^{2i\delta_l(k)}$$

which gives the ratio of the outgoing wave amplitude to the incoming wave amplitude (up to a complex factor of unit norm) $S_l(k) = A/B$ for the radial function of the asymptotic form as $r \to \infty$ (19.5.37, Pg 552):

$$R_{kl} \to \frac{Ae^{ikr}}{r} + \frac{Be^{-ikr}}{r}$$

The total cross section can be written as a sum of the individual cross sections at each l (19.5.19, Pg 548):

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$
$$\sigma = \sum_{l=0}^{\infty} \sigma_l$$

which turns out to be (19.5.21, Pg 548):

$$\sigma = \frac{4\pi}{k} \Im[f(0)]$$

which is called the **Optical theorem**.

Scattering resonances Let $\delta_l(k)$ rise rapidly near k_0 (or equivalently near E_0 (19.5.30, Pg 550):

$$\delta_l = \delta_b + \tan^{-1}\left(\frac{\Gamma/2}{E_0 - E}\right)$$

where δ_b is a slowly-varying background phase. Then the cross section for each angular momentum l can be written in **Breit-Wigner form** around $E \approx E_0$ (19.5.31, Pg 551):

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E_0 - E)^2 + (\Gamma/2)^2}$$

so that the half-width is $\Gamma/2$. Γ depends on k with (19.5.32, Pg 551):

$$\Gamma/2 = (kr_0)^{2l+1}\gamma$$

where r_0 is the characteristic range of the potential and γ is a constant with energy dimensions.

Two-particle scattering The rate of scattering events per volume of interaction for two beams of density ρ_1, ρ_2 moving with velocities v_1, v_2 is (19.6.2, Pg 556):

$$\sigma\rho_1\rho_2(v_1 + v_2) = \sigma\rho_1\rho_2 v_{\text{rel}}$$

Hence the differential cross section is defined based on the number of particles scattered into $d\Omega$ per volume of interaction (19.6.3, Pg 556):

$$\frac{d\sigma}{d\Omega} d\Omega \rho_1\rho_2 v_{\text{rel}}$$

The relationship between lab (L) and CM (unprimed) differential cross sections is (19.6.5, Pg 556):

$$\frac{d\sigma}{d\Omega_L} = \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\Omega_L}$$

Write the asymptotic solution in terms of a CM part and a relative part (19.6.9, Pg 557):

$$\psi \xrightarrow[r\to\infty]{} \psi^{CM}(z_{CM})\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$$

where $z_{CM} = (z_1 + z_2)/2, \psi^{CM} = e^{i(k_1+k_2)z_{CM}}$ and $k = (k_1 - k_2)/2$. The scattering cross section is hence (19.6.13, Pg 558):

$$\frac{d\sigma}{d\Omega} = |f|^2$$

Scattering angle relationship between lab and CM frames (19.6.17, Pg 559) where θ is the angle between the initial direction of the beam and the final direction.

$$\theta_L = \frac{\theta}{2}$$

Note: non-relativistic.

Scattering amplitude, identical particles (19.6.19, Pg 560) after symmetrizing the scattering wavefunction, the symmetric scattering amplitude is:

$$f_{\text{sym}}(\theta,\phi) = f(\theta,\phi) + f(\pi - \theta, \phi + \pi)$$

To find the total scattering cross section, integrate over 2π radians only.