Ph106a Class Notes

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Chapter 1

Week 1

1.1 Tuesday, 29 Sept 2015

1.1.1 Motivation and overview

Example 1 Consider a particle constrained to move along a curved surface in 3D. There will be three equations (3 components) along with an additional constraining equation that the particle is along the surface. This will be simplified to two equations (2 degrees of freedom). In analytical mechanics, it is not necessary to deal with the constraints directly.

Example 2 Rigid body motion: Constraints are such that the distance between particles does not change. Body has only 3 degrees of freedom.

Variational calculus Consider the curved surface problem. First consider the 4 dimensional space that describes the motion of the particle $(u, v, \dot{u}, \dot{v})$. Newton's laws will give the trajectory of the particle in the configuration space. Variational mechanics allows us to pick out the physical path from the other possible other paths - pick based on taking the extremum (usually minimum) of the action:

$$A = \int L(u, v, \dot{u}, \dot{v}) dt \tag{1.1}$$

For a curved surface, the physical path connecting any two points (under no other forces other than the constraint) will be the geodesic of the surface.

Example 3 Consider coupled harmonic oscillators on a ring. The strategy is to convert the N coupled equations in N decoupled normal modes.

Example 4 Consider central force (orbital) motion. We reduce the 3D equations into 2D ones by arguing that motion occurs along a plane, the decouple the two dimensions into radial and angular components.

Example 5 But some systems cannot be separated. Consider a central body that is almost spherical (spherical with bumps). Then the equation of motions of particles around this body will be the ideal equation with perturbations (such as precession).

Chaotic systems Cannot be separated. Perturbations in initial conditions result in an exponential increase in phase space occupancy so that in a finite amount of time the motion is difficult to predict.

1.1.2 Review of Newtonian Mechanics

Notation :

- Position: $\vec{r} = (x, y, z) \sim x\hat{i} + y\hat{j} + z\hat{k}$.
- Displacement: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$
- Velocity $\dot{\vec{r}} = \vec{v}$
- Acceleration $\dot{\vec{v}} = \vec{a}$.

Newton's Laws (1st) In an inertial frame, an object will not experience an acceleration (will move with constant velocity) in the absence of a net external force. (2nd) Relates force to acceleration. (3rd) $\vec{f}_{12} = -\vec{f}_{21}$. Leads to momentum conservation:

$$\vec{p} \equiv m\vec{v} \tag{1.2}$$

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1.3}$$

$$\vec{p}_{tot} = \sum_j \vec{p}_j \implies \frac{d\vec{p}_{tot}}{dt} = \sum_j \vec{F}_j = \sum_j \sum_{i\neq j} \vec{f}_{ij} = \frac{1}{2}\left(\sum_j \sum_{i\neq j} \vec{f}_{ij} + \sum_i \sum_{j\neq i} \vec{f}_{ij}\right) = \frac{1}{2}\sum_j \sum_{i\neq j} (\vec{f}_{ij} + \vec{f}_{ji}) \tag{1.4}$$

where $\vec{f}_{ij}$ is the force from the ith particle to the jth particle. Now implement Newton's 3rd law to ensure that the last statement (1.4) becomes zero.

Problems with N3L Consider two charged particles moving orthogonal to each other. The magnetic forces do not appear to satisfy N3L. Recall that the magnetic field is controlled by $\vec{r} \times \vec{v}$. Resolve this by considering that the forces act between the object and the field, not directly between objects. We need to introduce the field momentum.

Angular momentum Define:

$$\vec{L} \equiv \vec{r} \times \vec{p} \tag{1.5}$$

The rate of change is given by:

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} \tag{1.6}$$

and the first term vanishes because of the definition of momentum (momentum parallel to velocity).

Central force field is such that:

$$\vec{r} \times \vec{F} = 0 \implies \dot{\vec{L}} = 0 \tag{1.7}$$

because the force is always oriented towards the origin, which is parallel to the position vector. Hence the central force field conserves angular momentum. We can always draw the plane in configuration space that is orthogonal to the conserved angular momentum vector. Since $\vec{r} \times \vec{v} = \vec{L}$ and $\vec{L}$ is a constant, we have that $\vec{r} \perp \vec{L}$ and $\vec{v} \perp \vec{L}$. Hence the $\vec{r}, \vec{v}$ vectors lie in the plane orthogonal to the $\vec{L}$ vector, and we have reduced the 3D problem into a 2D problem with the constraint.

Origin displacement Angular momentum is defined with respect to a single point. It is possible to show that the definition of torque in terms of the rate of change of angular momentum is independent of origin position.

Energy and work Define:

$$W_{12} \equiv \int_1^2 \vec{F} \cdot d\vec{s} \tag{1.8}$$

for a particle moving from position 1 to position 2.

$$W_{12} = \int_1^2 m\dot{\vec{v}} \cdot \vec{v} dt \tag{1.9}$$

$$= \frac{1}{2} m \left[\vec{v} \cdot \vec{v}\right]_1^2 \quad \text{F.T.C} \tag{1.10}$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \tag{1.11}$$

and we define the kinetic energy $T = \frac{1}{2}mv^2$ accordingly.

Conservative forces are such that the work done between any two points does not depend on the path taken. Equivalently,

$$\oint \vec{F} \cdot d\vec{s} = 0 \tag{1.12}$$

Potential Since the conservative force work done is independent of path, we can always pick an arbitrary point to be the origin and then integrate the work done along that path to define the potential:

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{\vec{r}} \vec{F} \cdot d\vec{s} \tag{1.13}$$

$$W_{12} = W_{1\to\mathcal{O}} + W_{\mathcal{O}\to 2} = V(\vec{r}_1) - V(\vec{r}_2) \tag{1.14}$$

This can be combined with the expression for kinetic energy to give:

$$W_{12} = T_2 - T_1 \implies T_1 + V_1 = T_2 + V_2 \tag{1.15}$$

Force and Potential Energy Given a potential energy that is a single-valued function of position, we have:

$$\vec{F} = -\nabla V \tag{1.16}$$

This can be shown by considering the small displacement $(x, y, z) \to (x + \Delta x, y, z)$:

$$\Delta V = -\int_{(x,y,z)}^{(x+\Delta x,y,z)} dw F_x(x + w, y, z) \tag{1.17}$$

where we parametrize the path using the variable w. Taking the limit as $\Delta x \to 0$, we obtain:

$$\frac{\partial V}{\partial x} = -F_x \tag{1.18}$$

Repeat for other directions.

Geometric interpretation The force points in the direction opposite to that of greatest increase in potential.

1.2 Thursday, 1 Oct 2015

1.2.1 Newtonian Gravity

Gravitational Potential

$$\nabla^2\phi = 4\pi G\rho \tag{1.19}$$

$$\phi(\vec{r}) = \int \frac{G\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d^3\vec{r'} \tag{1.20}$$

$$V(\vec{r}) = m\phi(\vec{r}) \tag{1.21}$$

$$\vec{F} = -m\nabla\phi \tag{1.22}$$

$$\rho(\vec{r}) = M\delta^3(\vec{r} - \vec{r}_0) \tag{1.23}$$

Note that the mass in equation (1.21) is called the passive gravitational mass in that it passively experiences the gravitational field. On the contrary, the mass in equation (1.23) is called the active gravitational mass because it generates the gravitational field.

Near Earth's Surface take the acceleration to be a constant so that:

$$\phi = -\vec{g} \cdot \vec{r} + \text{const} \tag{1.24}$$

Clearly, if $\vec{g}$ is a constant,

$$\vec{F} = -\nabla\phi = \nabla(\vec{g} \cdot \vec{r}) = \nabla(g_x X + g_y Y + g_z Z) = (g_x, g_y, g_z) = \vec{g} \tag{1.25}$$

1.2.2 Example of constrained motion

Description Consider an object with mass m on an inclined surface of angle α.

Method I: Standard solution Resolve forces. Consider constraint explicitly: $\dot{y} = -\dot{x}\tan\alpha$.

Method II: Clever choice of coordinates We note that the system has only 1 degree of freedom. Hence we choose a coordinate to represent motion along that degree of freedom. Resolve gravity.

Method III: Energy balance Claim that energy is conserved:

$$\frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy = E_0 \tag{1.26}$$

Implement the constraint $\dot{y} = -\dot{x}\tan\alpha$. Combine to obtain ODE:

$$\frac{m}{2\sin^2\alpha}\dot{y}^2 + mgy = E_0 \tag{1.27}$$

Rearranging:

$$\dot{y} = \pm\sqrt{\frac{2(E_0 - mgy)\sin^2\alpha}{m}} \tag{1.28}$$

Note that the plus minus sign depends on the sign of the initial velocity $\dot{y}_0$. Hence we can write:

$$\dot{y} = \text{sgn}(\dot{y}_0)\sqrt{\frac{2(E_0 - mgy)\sin^2\alpha}{m}} \tag{1.29}$$

Integrating the ODE directly,

$$\int_{y_0}^{y(t)} \frac{dy}{\sqrt{2(E_0/m - gy)\sin^2\alpha}} = \int_0^t \text{sgn}(\dot{y}_0)dt' \tag{1.30}$$

$$\sqrt{2(E_0/m - gy)} = \sqrt{2(E_0/m - gy_0)} - \text{sgn}(\dot{y}_0)g\sin\alpha t \tag{1.31}$$

To remove the sign function, square both sides and note that the cross term $\text{sgn}(\dot{y}_0)|\dot{y}_0| = \dot{y}_0$. The second term is actually:

$$\sqrt{2(E_0/m - gy_0)} = |\dot{y}_0| \tag{1.32}$$

General form Note that the 1D COE equation can be written in the form:

$$\frac{1}{2}\dot{y}^2 + V(y) = E \tag{1.33}$$

where we normalize the mass. The ODE to solve is hence:

$$\dot{y} = \pm\sqrt{2(E - V(y))} \tag{1.34}$$

This is considered solved. But there is a subtle issue when $V(y) = E$ for some values of y. If the potential energy crosses the total energy with nonzero slope in y (so that you reach the intersection in a finite amount of time), we can always linearize around the intersection point:

$$E_0 - V = E_0 - V(y^*) - V'(y^*)(y - y^*) + O(y - y^*)^2 = V'(y^*)(y^* - y) + O(y - y^*)^2 \tag{1.35}$$

Then we can use analytic integrals to evaluate the motion around that point, and we will obtain something like $y \sim y^* - \frac{1}{2}V'(y^*)t^2$.

Potential energy touching total energy tangentially In such a case, the time taken for the particle to reach the intersection point is infinite. In this case, the first order Taylor expansion is insufficient because it vanishes at the point as well. Hence we need the second order:

$$E_0 - V = -\frac{1}{2}V''(y^*)(y - y^*)^2 + O(y - y^*)^3 \tag{1.36}$$

Now we have a quadratic term in the square root of the denominator. This integral is not convergent because it integrates to a logarithm. It takes an infinite amount of time for the particle to approach the turning point.

1.2.3 System of particles

Total momentum

$$\vec{P} = \sum_j \vec{p}_j \tag{1.37}$$

Under N3L, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\vec{P} = \vec{F}^{\text{ext}}$. By definition:

$$\vec{P} = \sum_j m_j \frac{d\vec{r}_j}{dt} = \frac{d}{dt}\left(\sum_j m_j \vec{r}_j\right) = M\frac{d}{dt}\left(\frac{\sum_j m_j \vec{r}_j}{\sum_j m_j}\right) \tag{1.38}$$

where M is the total mass $\sum_j m_j$. Define the center of mass $\vec{R}$, the mass weighted average of the positions:

$$\vec{R} = \left(\frac{\sum_j m_j \vec{r}_j}{\sum_j m_j}\right) \tag{1.39}$$

$$\implies \vec{P} = M\frac{d\vec{R}}{dt} \tag{1.40}$$

$$\implies M\ddot{\vec{R}} = \vec{R}^{\text{ext}} \tag{1.41}$$

Center of mass frame has an offset of $\vec{R}$ in position and has an acceleration $\ddot{\vec{R}}$ (so it is generally not inertial). Converting to the CM frame coordinate $\vec{r'}_j$:

$$\vec{r}_j = \vec{R} + \vec{r'}_j \tag{1.42}$$

Also, by definition of the center of mass, the CM of the system is at the origin:

$$\sum_j m_j \vec{r}_j = M\vec{R} \implies \sum_j m_j \vec{r'}_j = 0 \tag{1.43}$$

Angular momentum Recall that:

$$\vec{L} = \sum_j \vec{r} \times m_j \vec{v}_j \tag{1.44}$$

In the CM frame:

$$\begin{aligned}
\vec{L} &= \sum_j (\vec{R} + \vec{r'}_j) \times m_j(\vec{V} + \vec{v'}_j) && (1.45)\\
&= \sum_j \vec{R} \times m_j \vec{V} + \sum_j \vec{R} \times m_j \vec{v'}_j + \vec{r'}_j \times m_j \vec{V} + \vec{r'}_j \times m_j \vec{v'}_j && (1.46)\\
&= M\vec{R} \times \vec{P} + \vec{R} \times \sum_j m_j \vec{v'}_j + \sum_j m_j \vec{r'}_j \times \vec{V} + \sum_j m_j \vec{r'}_j \times \vec{v'}_j && (1.47)\\
&= M\vec{R} \times \vec{P} + \sum_j m_j \vec{r'}_j \times \vec{v'}_j && (1.48)\\
&= \vec{L}_{CM} + \vec{L}_{\text{rotation about CM}} && (1.49)
\end{aligned}$$

So that the rate of change is:

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \sum_j \vec{r}_j \times \dot{\vec{p}}_j + \sum_j \dot{\vec{r}}_j \times \vec{p}_j && (1.50)\\
&= \sum_j \vec{r}_j \times \dot{\vec{p}}_j && (1.51)\\
&= \sum_j \vec{r}_j \times \left(\vec{F}_j^{\text{ext}} + \sum_{i \neq j} \vec{F}_{ij} \right) && (1.52)\\
&= \vec{N}_{\text{ext}} + \sum_j \sum_{i \neq j} (\vec{r}_j \times \vec{F}_{ij}) && (1.53)\\
&= \vec{N}_{\text{ext}} + \frac{1}{2} \sum_j \sum_{i \neq j} (\vec{r}_j - \vec{r}_i) \times \vec{F}_{ij} && (1.54)
\end{aligned}$$

Hence for the internal torques to vanish, we require not only that the internal forces are equal and opposite, but also that they lie along the same line. Then we have $\frac{d\vec{L}}{dt} = \vec{N}_{\text{ext}}$ as expected. If the reaction forces do not lie on the same line, we can expect that the forces are the result of a field with non-zero angular momentum.

Considering the CM frame,

$$\begin{aligned}
\dot{\vec{L}} &= \vec{R} \times \dot{\vec{P}} + \dot{\vec{L'}} && (1.55)\\
\implies \sum_j \left(\vec{r}_j \times \vec{F}_j^{\text{ext}} - \vec{R} \times \vec{F}_j^{\text{ext}} \right) &= \sum_j \vec{r'}_j \times \vec{F}_j^{\text{ext}} = \dot{\vec{L'}} && (1.56)
\end{aligned}$$

and we recognize the term on the LHS to be the torque with respect to the CM.

Energy of a many-particle system Write the force equation as:

$$\vec{F}_j = -\nabla_j V_j - \sum_{i \neq j} \nabla_j V_{ij} \tag{1.57}$$

and assume that the potential has the form:

$$V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|) \tag{1.58}$$

then the energy is conserved:

$$\sum_j T_j + \sum_j V_j + \frac{1}{2} \sum_j \sum_{i \neq j} V_{ij} = \text{const} \tag{1.59}$$

Chapter 2

Week 2

2.1 Monday, 6 Oct 2015

2.1.1 Lagrangian Mechanics

Example 1: Block on inclined plane There are 6 equations of motion (2 vector equations and 2 constraint equations). Let N_1 be the normal force along the inclined plane and N_2 be the normal force on the ground. Then:

$$m\vec{g} + \vec{N}_1 = m\vec{a}_m \tag{2.1}$$

$$m\vec{g} - \vec{N}_1 + \vec{N}_2 = M\vec{a}_M \tag{2.2}$$

$$\vec{a}_M \cdot \hat{j} = 0 \tag{2.3}$$

$$(\vec{a}_m - \vec{a}_M) \cdot \hat{n} = 0 \tag{2.4}$$

Note that these 6 equations completely define the 6 unknowns (4x coordinates and 2x normal force magnitudes).

Example 1 using parametrized variables Define X and d to be the position of the inclined plane and block along the inclined plane respectively. Call the (X, d) space the configuration space. Defining:

$$\vec{a}_M = \ddot{X}\hat{i} \tag{2.5}$$

$$\vec{a}_m = \ddot{X}\hat{i} + \ddot{d}(\hat{i}\cos\alpha + \hat{j}\sin\alpha) \tag{2.6}$$

Inserting these equations into the 6 equations removes 2 equations (constraints are automatically satisfied) and removes 2 unknowns (4 variables parametrized by 2 variables). But we can do better.

Virtual displacement Consider the configuration space. Consider a displacement of the configuration by a small amount $(\delta X, \delta d)$. We want to know how much work is induced by this displacement (computed in the Euclidean space):

$$\delta W = \sum_i \vec{F}_i \cdot \delta\vec{r}_i \tag{2.7}$$

where we sum over all particles.

Under a dissipationless system, the constraint forces will not do any work. Hence:

$$\delta W = \sum_i \vec{F}_i^{\text{ext}} \cdot \delta\vec{r}_i \tag{2.8}$$

Now we also know that:

$$\delta W = \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i \tag{2.9}$$

The summations will result in a linear combination of perturbed variables $\delta X, \delta d$. Hence equating the two summations, we note that δX and δd are independent, so their coefficients must match on both sides of the equation.

We may further simplify the equation using the chain rule and some manipulation:

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial x_j} \delta x_j \tag{2.10}$$

where we sum over the parametrized variables in the configuration space instead of the number of particles.
Note further that for holonomic constraints:

$$\frac{\partial \vec{r}_i}{\partial x_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{x}_j} = \frac{\partial \vec{v}_i}{\partial \dot{x}_j} \tag{2.11}$$

Substituting this into the expression for the momentum summation:

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \dot{\vec{r}}_i \cdot \sum_j \frac{\partial \dot{\vec{r}}_i}{\partial \dot{x}_j} \delta x_j \tag{2.12}$$

We can write this in terms of derivatives of the total kinetic energy. See Lecture 3 notes if absolutely necessary. The end result is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}}\right) = \frac{\delta W}{\delta X} \tag{2.13}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{d}}\right) = \frac{\delta W}{\delta d} \tag{2.14}$$

Example 2: Bead on rotating rod Simplifies to:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = \frac{\delta W}{\delta q} \tag{2.15}$$

The kinetic energy is:

$$T = \frac{m}{2}\left(\dot{q} + (q \sin \alpha \omega)^2\right) \tag{2.16}$$

where α is the angle of the rod from the vertical. Substituting into the virtual work expression, we have:

$$LHS = m\ddot{q} - m\omega^2 \sin^2 \alpha q \tag{2.17}$$

the work done by the external force (gravity) is $-mg\cos\alpha$ hence the equation of motion is:

$$m\ddot{q} - m\omega^2 \sin^2 \alpha q = -mg\cos\alpha \tag{2.18}$$

Move into the rod non-inertial frame of reference. Then there is the vertical force of gravity, the centrifugal force outwards, $m(q \sin \alpha)\omega^2$, the Coriolis force $2m\vec{v} \times \vec{\omega}$. The Coriolis force is zero because $\vec{v}$ is orthogonal to the angular velocity vector (vertical). Resolving forces along the q direction, we will obtain the same equation of motion.

Note that even though we required the virtual work to be zero, the actual work is non-zero because the bead will move outwards and speed up (runaway process). The actual work is:

$$dW = \vec{N} \cdot d\vec{r} \tag{2.19}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q} dq + \frac{\partial \vec{r}}{\partial t} dt \tag{2.20}$$

Note that the virtual work does not consider the time term because we fix the time. Only the first term is relevant for virtual work. Hence:

$$dW = \left(\vec{N} \cdot \frac{\partial \vec{r}}{\partial q} \right) dq + \left(\vec{N} \cdot \frac{\partial \vec{r}}{\partial t} \right) dt \tag{2.21}$$

the first term is zero but the second term is non-zero.

2.2 Thursday, 8 Oct 2015

2.2.1 Notation

Holonomic system Let there be generalized coordinates $q_1, q_2, \ldots, q_N$ with N independent degrees of freedom. Let there be M particles described by $3M$ degrees of freedom $\vec{r}_1, \ldots, \vec{r}_M$. There is a map from the N degrees of freedom to the 3M coordinates in Euclidean space. Write this map as:

$$\vec{r}_j(q_1, \ldots, q_N, t) = \vec{r}_j(q_k, t) \tag{2.22}$$

where k is understood to run from $1, \ldots, N$.

We need $3M - N$ constraint equations:

$$f_l(\vec{r}_1, \ldots, \vec{r}_M, t) = 0, \quad l = 1, 2, \ldots, 3M - N \tag{2.23}$$

Using the identity derived using the rotating rod example from last lecture, we write the equation for each generalized coordinate as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = \frac{\delta W}{\delta q_k} \tag{2.24}$$

Derivation Start by writing the velocity of the jth object:

$$\vec{r}_j(q_k, t) \implies \vec{v}_j(q_k, \dot{q}_k, t) = \sum_k \frac{\partial \vec{r}_j}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_j}{\partial t} \tag{2.25}$$

For a holonomic system:

$$\frac{\partial \vec{v}_j}{\partial \dot{q}_k} = \frac{\partial \vec{r}_j}{\partial q_k}, \quad \forall j, k \tag{2.26}$$

Also observe that:

$$\frac{\partial \vec{v}_j}{\partial q_k} = \frac{\partial}{\partial q_k} \left(\sum_l \frac{\partial \vec{r}_j}{\partial q_l} \dot{q}_l + \frac{\partial \vec{r}_j}{\partial t} \right) = \sum_l \frac{\partial^2 \vec{r}_j}{\partial q_k \partial q_l} \dot{q}_l + \frac{\partial^2 \vec{r}_j}{\partial q_k \partial t} = \frac{d}{dt} \left(\frac{\partial \vec{r}_j}{\partial q_k} \right) \tag{2.27}$$

by the interchanging of partial derivatives.

Also consider the kinetic energy:

$$T = \sum_j \frac{m_j}{2} \vec{v}_j^2 \tag{2.28}$$

We first use d'Alembert's principle:

$$\sum_j \dot{\vec{p}}_j \cdot \delta \vec{r}_j = \delta W \tag{2.29}$$

We examine the LHS:

$$\begin{aligned}
\sum_j \dot{\vec{p}}_j \cdot \delta \vec{r}_j &= \sum_j m_j \dot{\vec{v}}_j \cdot \delta \vec{r}_j && (2.30)\\
&= \sum_j m_j \dot{\vec{v}}_j \cdot \sum_k \frac{\partial \vec{r}_j}{\partial q_k} \delta q_k && (2.31)\\
&= \sum_{j=1}^{M} \sum_{k=1}^{N} m_j \left(\dot{\vec{v}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k} \right) \delta q_k && (2.32)\\
&= \sum_{k=1}^{N} \left(\sum_{j=1}^{M} m_j \dot{\vec{v}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k} \right) \delta q_k \quad \text{exchanging order of summation} && (2.33)\\
&= \sum_{k=1}^{N} \left[\sum_{j=1} M \frac{d}{dt} m_j \vec{v}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k} - m_j \vec{v}_j \frac{d}{dt} \frac{\partial \vec{r}_j}{\partial q_k} \right] \delta q_k && (2.34)\\
&= \sum_{k=1}^{N} \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right] \delta q_k && (2.35)
\end{aligned}$$

Now examine the RHS:

$$\begin{aligned}
\delta W &= \sum_j \vec{F}_j \cdot \delta \vec{r} && (2.36)\\
&= \sum_j \vec{F}_j \cdot \left(\sum_{k=1}^{N} \frac{\partial \vec{r}_j}{\partial q_k} \delta q_k \right) && (2.37)\\
&= \sum_{k=1}^{N} \left(\sum_j \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k} \right) \delta q_k && (2.38)
\end{aligned}$$

Define the generalized force:

$$\mathcal{F}_k = \left(\sum_j \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_k} \right) \tag{2.39}$$

so that the RHS becomes:

$$\delta W = \sum_{k=1}^{N} \mathcal{F}_k \delta q_k \tag{2.40}$$

If the force is conservative, we have that:

$$\vec{F}_j = -\nabla_j V(\vec{r}_1, \ldots, \vec{r}_M) \tag{2.41}$$

Write the potential as a function of the generalized coordinates and time (because each of the Euclidean coordinates can depend on time):

$$V(q_1, \ldots, q_N, t) \tag{2.42}$$

We can simplify the generalized force accordingly using the chain rule:

$$-\frac{\partial V}{\partial q_k} = \sum_{j=1}^{M} -\frac{\partial V}{\partial \vec{r}_j} \cdot \frac{\partial \vec{r}_j}{\partial q_k} = \sum_{j=1}^{M} -\nabla_j V \cdot \frac{\partial \vec{r}_j}{\partial q_k} = \mathcal{F}_k \tag{2.43}$$

Hence we have:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \frac{\partial V}{\partial q_k} \tag{2.44}$$

If we consider that V does not depend on the velocities, then we have:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial \mathcal{L}}{\partial q_k} \tag{2.45}$$

$$\mathcal{L} = T - V \tag{2.46}$$

which is the Euler-Lagrange equation.

Define the generalized momentum:

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \tag{2.47}$$

so that:

$$\dot{p}_k = \frac{\partial \mathcal{L}}{\partial q_k} \tag{2.48}$$

Time derivative of Lagrangian Proceed by definition:

$$\dot{\mathcal{L}} = \sum_{k=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\ddot{q}_k \right) + \frac{\partial \mathcal{L}}{\partial t} \tag{2.49}$$

$$= \sum_{k=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\ddot{q}_k \right) \quad \text{kinetic and potential energy not explicitly dependent on time} \tag{2.50}$$

$$= \sum_{k=1}^{N} \left(\dot{p}_k \dot{q}_k + p_k \ddot{q}_k \right) \tag{2.51}$$

$$= \sum_{k=1}^{N} \frac{d}{dt} \left(p_k \dot{q}_k \right) \tag{2.52}$$

Hence:

$$\frac{d}{dt}\left(\sum_k p_k \dot{q}_k - \mathcal{L}\right) = -\frac{\partial L}{\partial t} = 0 \tag{2.53}$$

Hence the LHS term is conserved wrt time. Define this as the Hamiltonian:

$$\mathcal{H} = \left(\sum_k p_k \dot{q}_k - \mathcal{L}\right) \tag{2.54}$$

Note that it can be shown that:

$$\sum_k p_k \dot{q}_k = 2T \tag{2.55}$$

so that $\mathcal{H} = 2T - (T - V) = T + V$, the total energy in the conservative force case.

Chapter 3

Week 3

3.1 Tuesday 13 Oct 2015

Invariance implies conservation Recall that if $\frac{\partial L}{\partial t} = 0$, then we can use the Hamiltonian $H = \sum_k p_k \dot{q}_k - L$ to show that $\frac{dH}{dt} = 0$. Note the distinction between the total derivative and the partial derivative. Generally, $\frac{dL}{dt} \neq 0$.

When L does not depend on time explicitly, $q_k(t - t_0)$ is also a solution when $q_k(t)$ is a solution (because the Euler-Lagrange equation will only depend on q_k and $\dot{q}_k$).

Transational invariance Suppose L has a cyclic/ignorable coordinate q_0. Then $\dot{p}_0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} = 0$ using the Euler-Lagrange equations. This means that any solution displaced in q_0 remains a solution. We can also change variables to displace the coordinates in q_0 and the Lagrangian form remains identical.

Lagrangian T-V must be defined in inertial frame Consider a freely falling particle. Define a new accelerating frame $\tilde{y} = y - \frac{at^2}{2}$. In the inertial frame, the Lagrangian was just:

$$L(y, \dot{y}, t) = \frac{m\dot{y}^2}{2} - mgy \tag{3.1}$$

But in the non-inertial frame, we make the coordinate substitutions:

$$y = \tilde{y} + \frac{at^2}{2} \tag{3.2}$$

$$\dot{y} = \dot{\tilde{y}} + at \tag{3.3}$$

so that the Lagrangian is now dependent on time explicitly:

$$L(\tilde{y}, \dot{\tilde{y}}, t) = \frac{m}{2}\left(\dot{\tilde{y}} + at\right)^2 - mg\left(\tilde{y} + \frac{at^2}{2}\right) \tag{3.4}$$

Note that the kinetic and potential energies now pick up time-dependent terms, and hence are not simply the kinetic and potential energies in the accelerated frame. In order to write the Lagrangian as $T - V$, we must perform the measurements in an inertial frame.

Non-holonomic constraints: Example 1 (Differential but non-integrable constraint) Coin rolling without slipping on a plane. Let the position of the coin along the plane be (x, y) and let its velocity vector be pointing at the angle θ. The coin has an additional degree of freedom ϕ, the angle its plane makes with the vertical. There are two degrees of freedom: θ and ϕ which characterize the motion of the coin. The constraints are that the motion is rolling without slipping: $dx = Rd\phi\cos\theta, dy = Rd\phi\sin\theta$. Note that θ, ϕ are not sufficient to describe the state of the system. You need to know the entire variation of θ, ϕ in time to do this.

Non-holonomic constraints: Example 2 (Constraint conditions are inequalities) Ball falling off larger ball.

3.1.1 Calculus of variations

Consider the functional:

$$I[y] = \int_0^L F(y(x), y'(x), x)dx \tag{3.5}$$

Only consider functions whose endpoints are fixed. We want to extremize I[y] for functions with the fixed endpoints.

Example: Least-time problem Consider trajectory $y(x)$ under the influence of gravity. We require that $y(0) = 0$ and $y(L) = a$. The differential time is $dt = \frac{ds}{v} = \frac{ds}{\sqrt{2gy}}$.

Condition for extremum

$$I(y + \delta y) = I(y) + O(\delta y)^2 \tag{3.6}$$

We can introduce a perturbation ϵ so that:

$$I(y + \epsilon\delta y) = I(y) + O(\epsilon^2) \tag{3.7}$$

Hence we examine:

$$\begin{aligned} I(y + \epsilon\delta y) &= \int F(y(x) + \epsilon\delta y(x), y'(x) + \epsilon\delta y'(x), x)dx && (3.8) \\ &= I(y) + \int_0^L \epsilon \left.\frac{\partial F}{\partial y}\right|_x \delta y(x)dx + \int_0^L \epsilon \left.\frac{\partial F}{\partial y'}\right|_x \delta y'(x)dx + O(\epsilon^2) && (3.9) \\ &= I(y) + \int_0^L \epsilon \left.\frac{\partial F}{\partial y}\right|_x \delta y(x)dx + \left[\epsilon\frac{\partial F}{\partial y'}\delta y(x)\right]_0^L - \epsilon\int_0^L \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y(x)dx + O(\epsilon^2) && (3.10) \\ &= I(y) + \int_0^L \epsilon\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y(x)dx + O(\epsilon^2) && (3.11) \end{aligned}$$

Define the variational derivative:

$$\frac{\delta F}{\delta y(x)} \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \tag{3.12}$$

so that we can write:

$$\delta I = I(y + \delta y) - I(y) = \int_0^L \frac{\delta F}{\delta y(x)}\delta y dx + O(\delta y^2) \tag{3.13}$$

δI must vanish for all δy so that we have the extremum. This gives us the Euler equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{3.14}$$

Application to Lagrangian mechanics Consider the action:

$$S[q(t)] = \int_0^L L(q(t), \dot{q}(t), t)dt \tag{3.15}$$

We want to extremize the action, fixing the end points.

3.2 Thursday 15 Oct 2015

Curved Plane Let u, v parametrize the plane. Then the differential distance is:

$$ds^2 = g_{11}(u,v)du^2 + 2g_{12}(u,v)dudv + g_{22}(u,v)dv^2 \tag{3.16}$$

We want to compute the function giving the extremum distance between two points A and B. The distance functional is:

$$I = \int_0^L \sqrt{g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2} = \int_0^L dv\sqrt{g_{11}(u')^2 + 2g_{12}u' + g_{22}} \tag{3.17}$$

Hence the function inside the integral is $F(u', u, v) = \sqrt{g_{11}(u')^2 + 2g_{12}u' + g_{22}}$, and we can use Euler's equation to solve for the equation of the curve. These curves are called geodesics.

Note that is a unique geodesic for two points close by. But when the distance is increased, you may reach a conjugate point where uniqueness fails and there are many geodesics (think poles of the sphere). Conjugate points only occur in surfaces with positive curvature (spheres).

Example Consider the Lagrangian $L = \frac{m}{2}\dot{y}^2 - V(y)$. The variational term is (after integrating by parts and Taylor-expanding the potential):

$$\delta S = \int_0^t (-V' - m\ddot{y})\delta y dt + \int_0^t \left[\frac{m}{2}(\delta\dot{y})^2 - V''\frac{(\delta y)^2}{2}\right] dt + O[(\delta y)^3] \tag{3.18}$$

The first order term is just N2L, so it vanishes. Note that for the action to be a minimum, we require that the second term be positive. This is always fulfilled by $V'' > 0$. Also, we can estimate short time evolution by writing $T\delta\dot{y} \sim \delta y$ so that if $T < \sqrt{\frac{m}{V''}}$, then the second order coefficient is still positive and S is minimized for short times. See homework for details.

Change of variables Consider the change of variables from q_k space to Q_k space: $Q_k(q_k, t), \dot{Q}_k(q_k, \dot{q}_k, t)$. To operate in the Q_k space, we just replace the terms in the Lagrangian with the inverse map:

$$L(q_k(Q_k, t), \dot{q}_k(Q_k, \dot{Q}_k, t), t) \tag{3.19}$$

and use the Euler-Lagrange equations for Q_k accordingly. You will obtain the same path in the configuration space.

Non-holonomic constraints Consider the coin rolling on the surface example. The parameters x, y, θ, ϕ are not independent but are related by the constraints on their velocities:

$$\dot{x} = R\dot{\phi}\cos\theta \tag{3.20}$$

$$\dot{y} = R\dot{\phi}\sin\theta \tag{3.21}$$

We cannot use the Euler-Lagrange equation directly.

Review: Constrained optimization Suppose we want to extremize $F(x_1, \ldots, x_n)$ subject to $g_k(x_1, \ldots, x_n) = 0, k = 1, 2, \ldots, l$. We want to find the infinitesimal displacement $\delta\vec{x}$ such that:

$$\nabla F \cdot \delta\vec{x} = 0 \quad \text{first order condition for extremum} \tag{3.22}$$

$$\nabla\vec{g}_1 \cdot \delta\vec{x} = 0 \quad \text{constraint equations} \tag{3.23}$$

$$\vdots \tag{3.24}$$

$$\nabla\vec{g}_l \cdot \delta\vec{x} = 0 \tag{3.25}$$

This implies that ∇F lies in the space spanned by ∇g_k:

$$\nabla F = \sum_{k=1}^{l} -\lambda_k \nabla g_k \tag{3.26}$$

We may hence perform unconstrained optimization on the function:

$$F + \sum_k \lambda_k g_k \tag{3.27}$$

and require:

$$\nabla(F + \sum_k \lambda_k g_k) = 0 \tag{3.28}$$

$$g_1 = 0, \ldots, g_l = 0 \tag{3.29}$$

Example: Shape of hanging wire Consider a wire hung between points A and B. We want to find the configuration such that the potential energy is minimized:

$$V = \int_{-a}^{a} \rho g y ds \tag{3.30}$$

subject to the constraint that the total length of the wire is fixed:

$$l = \int_{-a}^{a} ds \tag{3.31}$$

We hence consider the function:

$$V + \lambda l \tag{3.32}$$

and optimize it using the Euler-Lagrange equation.

Example: Pendulum The constraint for a fixed length pendulum is:

$$g(x, y) = x^2 + y^2 - l^2 = 0 \tag{3.33}$$

and the Lagrangian is the usual $L = T(\dot{x}, \dot{y}) - V(y)$. Since we require that the constraint be fulfilled at all times, we consider the Lagrange multiplier problem:

$$\delta \left[S + \int_0^t \lambda(t') g(x(t'), y(t')) dt' \right] = 0 \implies \delta \int (L + \lambda g) dt = 0 \tag{3.34}$$

from which we have the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial g}{\partial x} = 0 \tag{3.35}$$

and the same for y. Substituting, we obtain the system:

$$-m\ddot{x} + 2\lambda x = 0 \tag{3.36}$$

$$-mg - m\ddot{y} + 2\lambda y = 0 \tag{3.37}$$

$$x^2 + y^2 = l^2 \tag{3.38}$$

which is 3 equations for 3 unknowns. We can solve this by linearizing around the bottom of the pendulum swing.

General holonomic constraints on coordinates and not velocities. Write:

$$L' = L + \sum_j \lambda_j g_j \tag{3.39}$$

and the new Euler-Lagrange equations are:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = -\sum_j \lambda_j \frac{\partial g_j}{\partial q_k} \tag{3.40}$$

Non-holonomic constraints and Lagrangians Consider the rolling coin on a plane again. Suppose the plane is angled at α with respect to the horizontal. Let x be horizontal and let y point along the plane towards the lower side of the plane. The Lagrangian is:

$$L = \frac{3}{4}mR^2(\dot{\phi})^2 + \frac{1}{8}mR^2(\dot{\theta})^2 + mgy\sin\alpha \tag{3.41}$$

where there are two forms of rotational motion with different moments of inertia for each. We include the constraints using the Lagrange multiplier method, using a Lagrange multiplier that is only dependent on time. See Lecture 6 notes for a full calculation.

Chapter 4

Week 4

4.1 Tuesday, 20 Oct 2015

Midterm Week 1-3 material. No Mathematica.

Review of previous weeks Recall that we noted that for the physical path, the action was stationary:

$$\delta S = 0 \iff \sum_k \frac{\delta L}{\delta q_k} \delta q_k = 0 \tag{4.1}$$

Given further constraints on q_k, we wrote these constraints as $G_j(q_k) = 0$ so that:

$$\sum_k \frac{\partial G_j}{\partial q_k} \delta q_k = 0 \tag{4.2}$$

For some non-holonomic constraints, the constraints were written as:

$$\sum_k f_k^{(j)} \delta q_k = 0 \tag{4.3}$$

In the holonomic case, we defined the new Lagrangian as satisfying:

$$\frac{\delta L}{\delta q_k} - \sum_j \lambda_j \frac{\partial G_j}{\partial q_k} = 0 \tag{4.4}$$

and for the non-holonomic special case, the equation was:

$$\frac{\delta L}{\delta q_k} - \sum_j \lambda_j f_k^{(j)} = 0 \tag{4.5}$$

4.1.1 Simple Harmonic Oscillator

Consider the simple pendulum:

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta \tag{4.6}$$

The equation of motion is:

$$ml^2 \ddot{\theta} = -mgl \sin\theta \tag{4.7}$$

Definition of equilibrium $\theta(t) = \theta_0$. Clearly, two such constant solutions are $\theta(t) = 0, \theta(t) = \pi$. We can perform a Taylor expansion of $V(\theta)$ around θ_0:

$$ml^2\ddot{\Delta\theta} = \pm mgl\Delta\theta \tag{4.8}$$

where the positive sign corresponds to $\theta_0 = \pi$ and the negative corresponds to $\theta_0 = 0$. Clearly, the $\theta_0 = 0$ case gives stable oscillations parametrized by $\sin\omega t, \cos\omega t$. The $\theta_0 = \pi$ case gives unstable motion that goes as $e^{\lambda t}, e^{-\lambda t}$.

Hamiltonian picture Recall that $H = T - V$. Then:

$$H = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{4.9}$$

and since the Lagrangian does not depend explicitly on time, the Hamiltonian is conserved. Plot the total energy as a function of θ to obtain the classically allowed region as a function of the initial energy.

General equilibrium Given a Lagrangian and Hamiltonian:

$$L = \frac{1}{2}\dot{q}^2 - V(q) \tag{4.10}$$

$$H = \frac{1}{2}\dot{q}^2 + V(q) \tag{4.11}$$

then the equilibrium condition is given by:

$$\frac{\partial V}{\partial q} = 0 \tag{4.12}$$

Stable and unstable equilibrium evaluated at the equilibrium point can be written as:

$$\frac{\partial^2 V}{\partial q^2} > 0 \implies \text{stable equilibrium} \tag{4.13}$$

$$\frac{\partial^2 V}{\partial q^2} < 0 \implies \text{unstable equilibrium} \tag{4.14}$$

Consider the Taylor expansion of the Lagrangian $L(q, \dot{q})$. The most general second order expansion about $(q, \dot{q}) = (0, 0)$ is:

$$L = A + Bq + C\dot{q} + Dq^2 + Eq\dot{q} + F\dot{q}^2 + \ldots \tag{4.15}$$

But we know that any time derivative can be added to the Lagrangian without changing the physical motion. Hence the $C\dot{q}$ and $Eq\dot{q} = \frac{E}{2}\frac{d}{dt}q^2$ terms are not important. Then the Lagrangian looks like the sum of a kinetic energy ($F\dot{q}^2$) and potential energy $A + Bq + Dq^2$. The Euler Lagrange equation gives the equation of motion:

$$2F\ddot{q} = 2Dq + B \tag{4.16}$$

For an equilibrium, we require $B = 0$ because we want $\frac{\partial V}{\partial q} = 0$ at the equilibrium. Hence the stability of an equilibria depends on the relative signs of F and D. If they are the same sign, the system is unstable about that point because the solution is an exponential. If they have opposite sign, the system is stable about that point because the solutions are sinusoids. We rewrite the equation of motion:

$$\ddot{q} = \frac{D}{F}q \tag{4.17}$$

To rescale the equation, define $\tau = \beta t$. Then we have the equation:

$$\frac{d^2 q}{d\tau^2} = \beta^2 \frac{D}{F} q \tag{4.18}$$

We hence want to choose $\beta = \sqrt{\left|\frac{F}{D}\right|}$ so that we can write:

$$\frac{d^2 q}{d\tau^2} = \pm q \tag{4.19}$$

where the sign depends on the stability of the equilibrium point. The action can be written in non-dimensionalized form as:

$$S = \int L dt = \beta \int L d\tau \tag{4.20}$$

Now we know that the Lagrangian is invariant under scaling, hence the action can be written as $S = \int L d\tau$. We can also neglect the A term in the Lagrangian because it does not affect the physical system. The action can hence be written as:

$$S = \int \left(\frac{|D|}{|F|} F \left(\frac{dq}{d\tau} \right)^2 + D q^2 \right) d\tau \tag{4.21}$$

Rescaling this equation further,

$$S = \int \left(\frac{F}{|F|} \left(\frac{dq}{d\tau} \right)^2 + \frac{D}{|D|} q^2 \right) d\tau \tag{4.22}$$

Hence the rescaled action is given by:

$$S = \int \frac{1}{2} \left[\left(\frac{dq}{d\tau} \right)^2 \pm q^2 \right] d\tau \tag{4.23}$$

where the plus represents the unstable equilibrium and the negative represents stable equilibrium.

Damped harmonic oscillator

$$\ddot{q} + q = -\frac{\dot{q}}{Q} \tag{4.24}$$

Free evolution of the DHO The characteristic equation is:

$$-\alpha^2 + i\alpha \frac{1}{Q} + 1 = 0 \tag{4.25}$$

$$\implies \alpha = \frac{i}{2Q} \pm \sqrt{1 - \frac{1}{4Q^2}} \tag{4.26}$$

Root locus for DHO For large Q, roots are located at ± 1 on the real axis. As Q decreases, the roots move towards the positive imaginary axis and touches when $Q = \frac{1}{2}$. As Q decreases further, the roots move further apart on the imaginary axis. One root goes to zero and the other goes to infinity.

Green's function approach to forced damped harmonic oscillator First solve the impulse response of the oscillator. That is, we want to find G such that:

$$\ddot{G}(t,t') + \frac{\dot{G}(t,t')}{Q} + G(t,t') = \delta(t-t') \tag{4.27}$$

Clearly, $G(t,t') = 0$ for $t < t'$. For later time, we write the solution as a linear combination of the two orthogonal functions.

Note that G has to be continuous at $t = t'$. Otherwise, $\dot{G}$ contains a delta function, and $\ddot{G}$ has a derivative of the delta function. This is not the case. Hence G must be continuous at $t = t'$. We integrate the ODE about the point $t = t'$:

$$\int_{t'-\epsilon}^{t'+\epsilon} \left[\ddot{G}(t,t') + \frac{\dot{G}(t,t')}{Q} + G(t,t') \right] dt = 1 \tag{4.28}$$

This simplifies to:

$$\dot{G}\Big|_{t'-\epsilon}^{t'+\epsilon} + \frac{G}{Q}\Big|_{t'-\epsilon}^{t'+\epsilon} + \int_{t'-\epsilon}^{t'+\epsilon} G dt = 1 \tag{4.29}$$

Now G is continuous, hence the second and third term vanishes when $\epsilon \to 0$. Hence we have the condition:

$$G(t'-0) = G(t'+0) \tag{4.30}$$
$$\dot{G}(t'+0) = \dot{G}(t'-0) + 1 \tag{4.31}$$

This provides the initial velocity of the system, which we can use to find G by plugging it into the normal solution. For instance, in the underdamped solution:

$$G(t,t') = \begin{cases} \frac{\sin \omega'(t-t')}{\omega'} e^{-(t-t')/2Q}, & t > t' \\ 0, \quad t \leq t' \end{cases} \tag{4.32}$$

where $\omega' = \sqrt{1 - \frac{1}{4Q^2}}$.

We want to use the Green's function to construct the particular solution to the ODE. This can be shown to be:

$$q_p(t) = \int_{-\infty}^{\infty} G(t-t')F(t')dt' = \int_{t_0}^{t} G(t-t')F(t')dt' \tag{4.33}$$

Observe that this is the steady state solution to the problem because the homogeneous solution decays away in time. If $t - t_0 >> 2Q$, the transient is not important and we can extend the integral to minus infinity:

$$q_p(t) = \int_{-\infty}^{t} dt' G(t-t')F(t') \tag{4.34}$$

Note that we have decomposed the solution into the complementary portion (solution to the homogeneous equation that satisfies the initial conditions $q_0, \dot{q}_0$) and the particular solution (solution that takes into account the RHS and has initial conditions that are zero).

4.2 Thursday 22 Oct 2015

Review Recall that with suitable scaling of time, we can always write the Lagrangian in the form:

$$L = \frac{1}{2}\dot{q}^2 - \frac{q}{2} \tag{4.35}$$

Fourier transform method Write:

$$q(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\tilde{q}(\omega) \tag{4.36}$$

and taking the Fourier transform of both sides of the ODE:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(-\omega^2 + \frac{i\omega}{Q} + 1\right)\tilde{q}(\omega)e^{i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\tilde{F}(\omega)e^{i\omega t} \tag{4.37}$$

so we can obtain the Fourier transform of the solution:

$$\tilde{q}(\omega) = \frac{\tilde{F}(\omega)}{-\omega^2 + i\omega/Q + 1} \tag{4.38}$$

$$\implies \tilde{q}(\omega) = \tilde{G}(\omega)\tilde{F}(\omega) \tag{4.39}$$

because the Fourier transform of $\delta(t)$ is a constant.

Resonance Take $\omega = 1$. Then the steady state solution can be written as:

$$q(t) = \int_0^t \sin(t - t')F(t')dt = \sin t \int_0^t \cos t' F(t')dt' - \cos t \int_0^t \sin t' F(t')dt' \tag{4.40}$$

Observe that if $F(t')$ contains a component that is exactly in phase with any of the $\sin t, \cos t$ components, there will be an integrated component that increases in time, indicating resonance. In steady state, we write:

$$F = F_0 e^{i\omega t} \tag{4.41}$$

$$q = q_0 e^{i\omega t} \tag{4.42}$$

which allows us to obtain q_0 by substitution into the ODE:

$$q_0 = \frac{F_0}{-\omega^2 + i\omega/Q + 1} \tag{4.43}$$

The denominator does not vanish for finite Q. First consider $Q >> 1$. The total energy goes as $|q_0|^2$. This can be approximated near resonance $\omega \approx 1$:

$$|q_0|^2 = \frac{|F_0|^2}{(\omega^2 - 1)^2 + \omega^2/Q^2} \approx \frac{|F_0|^2}{4(\omega - 1)^2 + 1/Q^2} \tag{4.44}$$

This curve shape is called a Lorentzian. The half-maximum occurs when $\omega = 1 \pm \frac{1}{2Q}$ so that the full frequency width at half-maximum is $\frac{1}{Q}$. With units:

$$\frac{\Delta\omega_{FWHM}}{\omega} = \frac{1}{Q} \tag{4.45}$$

The phase of q_0 can be written as:

$$\phi = \arg\left[\frac{1}{-\omega^2 + i\omega/Q + 1}\right] = \arg(1 - \omega^2 - i\omega/Q) \tag{4.46}$$

Observe that for small ω, the argument is zero. As ω increases, the argument decreases below zero, reaches $-\frac{\pi}{2}$ upon resonance, then asymptotes to $-\pi$ as $\omega \to \infty$. Call $-\phi$ the phase lag, because $q(t)$ goes as $\cos(t + \phi)$.

Theory of small vibrations Consider the two coupled pendulums. The Lagrangian can be parametrized by the two angles:

$$L = \left(\frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)\right) - \left[\frac{mgl}{2}(\theta_1^2 + \theta_2^2) + \frac{kl^2}{2}(\theta_2 - \theta_1)^2\right] \tag{4.47}$$

The EL equations gives:

$$\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{k}{4m}(\theta_1 - \theta_2) = 0 \tag{4.48}$$

$$\ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{k}{4m}(\theta_2 - \theta_1) = 0 \tag{4.49}$$

Non-dimensionalizing the equation, we define:

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{4.50}$$

$$\eta = \frac{kl}{4mg} \tag{4.51}$$

so we have the equations:

$$\ddot{\theta}_1 + \omega_0^2\theta_1 + \omega_0^2\eta(\theta_1 - \theta_2) = 0 \tag{4.52}$$

$$\ddot{\theta}_2 + \omega_0^2\theta_2 + \omega_0^2\eta(\theta_2 - \theta_1) = 0 \tag{4.53}$$

We hence have the matrix equation:

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} + \omega_0^2\begin{pmatrix}1+\eta & -\eta \\ -\eta & 1+\eta\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = 0 \tag{4.54}$$

We hence want to obtain the normal modes that satisfy:

$$\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}A_1 \\ A_2\end{pmatrix}e^{i\omega t} \tag{4.55}$$

which oscillates with a single frequency. Substitution into the matrix differential equation gives:

$$-\omega^2\boldsymbol{I}\begin{pmatrix}A_1 \\ A_2\end{pmatrix} + \omega_0^2\boldsymbol{M}\begin{pmatrix}A_1 \\ A_2\end{pmatrix} = 0 \tag{4.56}$$

which indicates that ω^2 is an eigenvalue of $\omega_0^2\boldsymbol{M}$ and the A vector is an eigenvector of $\omega_0^2\boldsymbol{M}$. Define $\omega^2 = \lambda\omega_0^2$, which gives the eigenvalues:

$$\lambda_1 = 1 \quad \lambda_2 = 1 + 2\eta \tag{4.57}$$

and the eigenvectors are:

$$v_1 = \begin{pmatrix}1 \\ 1\end{pmatrix} \tag{4.58}$$

$$v_2 = \begin{pmatrix}1 \\ -1\end{pmatrix} \tag{4.59}$$

4.3 Lagrangian of stretched string

Consider a small segment of string.

$$dT = \frac{1}{2}\lambda dx\dot{y}^2 \tag{4.60}$$

$$dV \approx F(ds - dx) = F\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - 1\right)dx \approx \frac{1}{2}Fy'^2 dx \tag{4.61}$$

So the action is given by:

$$S = \int_0^t \left[\int_{x_a}^{x_b} dx(\frac{1}{2}\lambda\dot{y}^2 - \frac{1}{2}Fy'^2)\right] \tag{4.62}$$

Note that the Lagrangian is given by $y, \dot{y}, y', t$, which is different from the usual Lagrangian. The same action minimizing principle in δy applies. Consider the Lagrangian density:

$$\delta S = \int dtdx\left(\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{\partial L}{\partial \dot{y}}d\dot{y}\right) \tag{4.63}$$

We integrate the second term by parts and eliminate the boundary term. Also integrate the third term by parts and cancel the boundary term. The final equation is:

$$\frac{\partial L}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \tag{4.64}$$

Applying this to the string, we obtain the equation of motion:

$$F\frac{\partial^2 y}{\partial x^2} = \lambda\frac{\partial^2 y}{\partial t^2} \tag{4.65}$$

Chapter 5

Week 5

5.1 Tuesday 27 Oct 2015

Theory of small vibrations - General Write the equation of motion in terms of a matrix DE:

$$\frac{d^2}{dt^2}\boldsymbol{x} + \boldsymbol{M}\boldsymbol{x} = 0 \tag{5.1}$$

where $\boldsymbol{M} = \boldsymbol{M}^T$ is symmetric. The eigenvalues are the resonant frequencies ω and the eigenvectors are the normal modes $\boldsymbol{A}$. The general solution is:

$$\boldsymbol{x}(t) = \Re\left(\sum_{j=1}^{N} C_j \boldsymbol{A}_j e^{i\omega_j t}\right) \tag{5.2}$$

where $C_j \in \mathbb{C}$.

Kinetic and Potential energies Let the system have kinetic and potential energies of the form:

$$T = \sum_{j,k} \dot{\phi}_j T_{j,k} \dot{\phi}_k \tag{5.3}$$

$$V = \sum_{j,k} \phi_j V_{jk} \phi_k \tag{5.4}$$

which can be obtained using Taylor expansion and keeping the second order terms alone. The matrix terms are:

$$T_{j,k} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{\phi}_j \partial \dot{\phi}_k} \tag{5.5}$$

$$V_{j,k} = \frac{1}{2}\frac{\partial^2 V}{\partial \phi_j \partial \phi_k} \tag{5.6}$$

Note that we can write these energies in the matrix quadratic form:

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \tag{5.7}$$

$$\boldsymbol{\phi}^T = (\phi_1, \dots, \phi_N) \tag{5.8}$$

$$\boldsymbol{T} = \{T_{j,k}\}_{j,k=1}^{N} = \boldsymbol{T}^T \quad \text{symmetric} \tag{5.9}$$

$$\boldsymbol{V} = \{V_{j,k}\}_{j,k=1}^{N} = \boldsymbol{V}^T \quad \text{symmetric} \tag{5.10}$$

$$T = \dot{\boldsymbol{\phi}}^T \boldsymbol{T} \dot{\boldsymbol{\phi}} \tag{5.11}$$

$$V = \boldsymbol{\phi}^T \boldsymbol{V} \boldsymbol{\phi} \tag{5.12}$$

Euler-Lagrange equations in matrix form for Small Vibrations Let $l = 1, 2, \ldots, N$.

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_l} &= \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}_l}\left(\sum_{j,k} T_{j,k}\dot{\phi}_j\dot{\phi}_k\right) && (5.13)\\
&= \frac{d}{dt}\sum_{j,k} T_{j,k}\left(\delta_{kl}\dot{\phi}_j + \dot{\phi}_k\delta_{jl}\right) && (5.14)\\
&= \frac{d}{dt}\left[\sum_j T_{lj}\dot{\phi}_j + \sum_k T_{kl}\dot{\phi}_k\right] && (5.15)\\
&= \frac{d}{dt}\left[\sum_j T_{lj}\dot{\phi}_j + \sum_k T_{lk}\dot{\phi}_k\right], \quad \mathbf{T} \text{ is symmetric} && (5.16)\\
&= 2\sum_j T_{lj}\ddot{\phi}_j && (5.17)\\
\implies \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{\phi}}} &= 2\boldsymbol{T}\ddot{\boldsymbol{\phi}} && (5.18)
\end{aligned}$$

Proceed similarly for the potential to obtain:

$$\begin{aligned}
\frac{\partial L}{\partial \phi_l} &= -\frac{\partial V}{\partial \phi_l} = -2\sum_j V_{lj}\phi_j && (5.19)\\
\implies \frac{\partial L}{\partial \boldsymbol{\phi}} &= -2\boldsymbol{V}\boldsymbol{\phi} && (5.20)
\end{aligned}$$

Implementing the Euler-Lagrange equation:

$$\boldsymbol{T}\ddot{\boldsymbol{\phi}} + \boldsymbol{V}\boldsymbol{\phi} = \mathbf{0} \tag{5.21}$$

where $\boldsymbol{T}, \boldsymbol{V}$ are $N \times N$ symmetric, positive definite, real-valued matrices:

$$\begin{aligned}
\boldsymbol{T} &= \boldsymbol{T}^T && (5.22)\\
\boldsymbol{V} &= \boldsymbol{V}^T && (5.23)\\
\boldsymbol{\phi}^T\boldsymbol{T}\boldsymbol{\phi} \geq 0, &\quad \boldsymbol{\phi}^T\boldsymbol{V}\boldsymbol{\phi} \geq 0 && (5.24)
\end{aligned}$$

where equality holds only when $\boldsymbol{\phi} = \mathbf{0}$. Note that while the kinetic energy is always positive anyway, the potential positive definite condition requires that the potential increase in all directions away from the equilibrium point. This is valid for small oscillations.

Solution to E-L equations using Normal Mode Approach Look for solutions in the form:

$$\begin{aligned}
\boldsymbol{\phi} &= \boldsymbol{A}e^{i\omega t} && (5.25)\\
\implies -\omega^2\boldsymbol{T}\boldsymbol{A} + \boldsymbol{V}\boldsymbol{A} &= \mathbf{0} && (5.26)\\
\implies (-\omega^2\boldsymbol{T} + \boldsymbol{V})\boldsymbol{A} &= \mathbf{0} && (5.27)
\end{aligned}$$

Results from Linear Algebra Linear Algebra says that two positive definite matrices can be simultaneously diagonalized. That is, there exists a set of vectors $\{\mathbf{\Phi}_j\}, j = 1, 2, \ldots, N$ such that:

$$\begin{aligned}
\mathbf{\Phi}_j^T\boldsymbol{T}\mathbf{\Phi}_k &= \delta_{jk} && (5.28)\\
\mathbf{\Phi}_j^T\boldsymbol{V}\mathbf{\Phi}_k &= v_j\delta_{jk} && (5.29)
\end{aligned}$$

where v_j has units of potential energy divided by kinetic energy, or inverse time squared. The vectors have units of inverse square roots of energy. Note that $\boldsymbol{\Phi}_j^T \boldsymbol{T} \boldsymbol{\Phi}_k$ can be defined as the inner product of $(\boldsymbol{\Phi}_j^T, \boldsymbol{\Phi}_k)$ such that the vectors are orthonormal. Note that the $\boldsymbol{\Phi}$ are chosen so that the "eigenvalues" under the transformation are all unity and dimensionless. Substituting these normal modes into the E-L matrix equation,

$$\boldsymbol{\Phi}_j^T(-\omega^2 \boldsymbol{T} + \boldsymbol{V})\boldsymbol{\Phi}_k = \mathbf{0} \quad \forall j, k \tag{5.30}$$

$$\implies (-\omega^2 + v_j)\delta_{jk} = 0 \quad \forall j \tag{5.31}$$

$$\implies \omega_k^2 = v_k \tag{5.32}$$

Note further that without degeneracy, rescaled normal modes associated with different frequencies are orthogonal using the definition of the inner product as similar to that above:

$$(\boldsymbol{\Phi}_j, \boldsymbol{\Phi}_k) \equiv \boldsymbol{\Phi}_j^T \boldsymbol{T} \boldsymbol{\Phi}_k = \delta_{kj} \tag{5.33}$$

and we can obtain the frequencies using the rescaled normal modes:

$$\boldsymbol{\Phi}_k^T(-\omega_k^2 \boldsymbol{T} + \boldsymbol{V})\boldsymbol{\Phi}_k = -\omega_k^2 + \boldsymbol{\Phi}_k^T \boldsymbol{V} \boldsymbol{\Phi}_k \tag{5.34}$$

$$\implies \omega_k^2 = \boldsymbol{\Phi}_k^T \boldsymbol{V} \boldsymbol{\Phi}_k \tag{5.35}$$

Example: Triatomic molecule Let two springs of constant k connect a central molecule M to two side molecules of mass m. Let the displacements of the molecules be x_1, x_2, x_3 from one side to other respectively. Then we write the energies:

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2 \tag{5.36}$$

$$V = \frac{k}{2}\left[(x_3 - x_2)^2 + (x_2 - x_1)^2\right] \tag{5.37}$$

Writing down the matrix elements explicitly:

$$\boldsymbol{T} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.38}$$

$$\boldsymbol{V} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{5.39}$$

$$r \equiv \frac{M}{m} \tag{5.40}$$

We require:

$$(-\omega^2 \boldsymbol{T} + \boldsymbol{V})\boldsymbol{\Phi} = 0 \tag{5.41}$$

Non-dimensionalizing, we define $\lambda = \frac{m\omega^2}{k}$ and look for the values of λ such that $(-\lambda \boldsymbol{T} + \boldsymbol{V})$ has zero determinant. This gives $\lambda = 0, 1, 1 + \frac{2}{r}$.

5.2 Thursday 29 Oct 2015

5.2.1 Review of Linear Algebra

Congruent transformation Start with a symmetric matrix $\boldsymbol{A}$ and define the congruent transformation:

$$\boldsymbol{A} \to \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} \tag{5.42}$$

$\boldsymbol{A}$ can always be transformed into a matrix with +1s or -1s or 0s along the diagonal. A positive definite matrix $\boldsymbol{A}$ can be transformed into the identity matrix (all 1s along the diagonal). The eigenvalues of $\boldsymbol{A}$ are not conserved under a congruent transformation.

Small oscillation Lagrangian Recall that we could write the Lagrangian in the quadratic form as (tilde means transpose):

$$L = \tilde{\dot{\phi}}\boldsymbol{t}\dot{\phi} + \tilde{\phi}\boldsymbol{v}\phi \tag{5.43}$$

where ϕ is a column vector of generalized coordinates.

The Euler Lagrange equation requires that:

$$\boldsymbol{t}\ddot{\phi} + \boldsymbol{v}\phi = 0 \tag{5.44}$$

which has solutions that look like:

$$\phi = \Phi e^{i\omega t} \tag{5.45}$$

and the matrix equation for the normal modes Φ is:

$$(-\omega^2\boldsymbol{t} + \boldsymbol{v})\Phi = \boldsymbol{0} \tag{5.46}$$

and for a non-trivial Φ, we required that:

$$\det(-\omega^2\boldsymbol{t} + \boldsymbol{v}) = 0 \tag{5.47}$$

which is an Nth order polynomial in ω^2. The solutions (indexed by the associated eigenvalues ω_k^2) can be scaled to satisfy the orthonormal relations:

$$\tilde{\Phi}_k\boldsymbol{t}\Phi_j = \delta_{jk} \tag{5.48}$$

$$\tilde{\Phi}_k\boldsymbol{v}\Phi_j = v_k\delta_{jk} \tag{5.49}$$

where we define the inner product:

$$(\Phi_k, \Phi_j) = \tilde{\Phi}_k\boldsymbol{t}\Phi_j \tag{5.50}$$

and the rescaling condition:

$$\Phi_k \to \frac{\Phi_k}{\sqrt{(\Phi_k, \Phi_k)}} \tag{5.51}$$

Initial value condition Suppose we know the initial positions and velocities $\phi_0, \dot{\phi}_0$. Then the time evolution of the initial condition can be written in terms of a complex superposition of normal modes:

$$\phi(t) = \Re\left[\sum_{k=1}^{N} A_k\Phi_k e^{i\omega_k t}\right] \tag{5.52}$$

where $A_k \in \mathbb{C}$. At $t = 0$:

$$\phi(0) = \Re\left[\sum_{k=1}^{N} A_k\Phi_k\right] = \sum_k (\Re A_k)\Phi_k \tag{5.53}$$

$$\dot{\phi}(0) = -\sum_k \omega_k(\Im A_k)\Phi_k \tag{5.54}$$

where we note that the normal modes are real-valued. This gives a system of 2N linear relations so that we can obtain the real and imaginary parts of A_k to obtain the time evolution of the oscillator. We use the distributivity of the inner product and orthogonality condition to obtain the real and imaginary parts of A_k:

$$(\Phi_j, \phi_0) = \left(\Phi_j, \sum_k (\Re A_k)\Phi_k\right) = \sum_k (\Re A_k)(\Phi_j, \Phi_k) = \sum_k (\Re A_k)\delta_{jk} = \Re A_j \tag{5.55}$$

$$(\Phi_j, \dot{\phi}_0) = -\omega_j(\Im A_j) \tag{5.56}$$

New generalized coordinates: Normal Coordinates Note that we can write any oscillation as a linear combination of normal modes:

$$\phi = \sum_j z_j \Phi_j \iff z_j = (\Phi_j, \phi) \tag{5.57}$$

$$\dot{\phi} = \sum_j \dot{z}_j \Phi_j \iff \dot{z}_j = (\Phi_j, \dot{\phi}) \tag{5.58}$$

so z_j and $\dot{z}_j$ can be seen as a set of new generalized coordinates and velocities. Call these **Normal Coordinates**. But this means that:

$$T = \tilde{\dot{\phi}} \boldsymbol{t} \dot{\phi} = \left(\sum_j \dot{z}_j \tilde{\Phi}_j\right) \boldsymbol{t} \left(\sum_k \dot{z}_k \Phi_k\right) = \sum_{j,k} \dot{z}_j \dot{z}_k \delta_{jk} = \sum_j \dot{z}_j^2 \tag{5.59}$$

and similarly,

$$V = \sum_j v_j z_j^2 \tag{5.60}$$

Hence in this generalized coordinate space, the Lagrangian is simple:

$$L = \sum_j \left(\dot{z}_j^2 - v_j z_j^2\right) \tag{5.61}$$

and the Euler-Lagrange equations require that:

$$\ddot{z}_j = -v_j z_j \tag{5.62}$$

Natural Unit System The systematic approach is to find quantities that have different dimensions and are independent (cannot achieve a quantity that is dimensionless by taking products of the units).

Coupled Oscillators Consider N masses on a string clamped down between fixed ends. Let the tension in the string be τ and let the separation between beads be d. The kinetic and potential energies are calculated accordingly in the small amplitude approximation:

$$T = \frac{m}{2} \sum_j \dot{y}_j^2 \tag{5.63}$$

$$V = \sum_{k=0}^{N} \tau \left[\sqrt{d^2 + (y_{k+1} - y_k)^2} - d\right] \approx \frac{\tau}{2d} \sum_{k=0}^{N} (y_{k+1} - y_k)^2 \tag{5.64}$$

where we define $y_0 = y_{N+1} = 0$ as the zero displacement clamps. In matrix form, the $\boldsymbol{t}$ matrix is diagonal, but the $\boldsymbol{v}$ matrix is not. In fact, the $\boldsymbol{v}$ matrix has 2s along the diagonal and -1s along the super and sub diagonal (multiplied by a scaling constant).

We change the units by setting $m = \tau = d = 1$. Applying the Euler-Lagrange equation:

$$\ddot{y}_k = y_{k-1} - 2y_k + y_{k+1}, \quad k = 1, 2, \ldots, N \tag{5.65}$$

We can make a guess as to the form of the mode. We make a standing wave discrete Fourier transform ansatz:

$$y_k = \sum_{n=1}^{N} a_n \sin \frac{\pi n k}{N+1} e^{-i\omega_n t} \tag{5.66}$$

so that when $k = 0$ or $k = N + 1$, then y_k vanishes. We want to ensure that each of the basis terms satisfies the Euler Lagrange equation. Write a single term as:

$$y_k = e^{-i\omega_n t} \sin \frac{\pi n k}{N+1} = e^{-i\omega_n t} \sin \gamma_n k, \quad \gamma_n \equiv \frac{\pi n}{N+1} \tag{5.67}$$

Substituting into the E-L equations, we want:

$$-\omega_n^2 e^{i\omega_n t} \sin \gamma_n k = 2 \sin(\gamma_n k)(\cos \gamma_n - 2) e^{i\omega_n t} \tag{5.68}$$

$$-\omega_n^2 = 2(\cos \gamma_n - 1) \tag{5.69}$$

$$\implies \omega_n = \pm 2 \sin \frac{\gamma_n}{2} = \pm 2 \sin \frac{n\pi}{2(N+1)} \tag{5.70}$$

Plotting ω_n against n, we get a discrete function that increases like a sine from $n = 1$ to $n = N$, and peaks at a maximum value of 2. The phase and group velocities are as follows:

$$v_p = \frac{\omega}{\gamma} \tag{5.71}$$

$$v_g = \frac{d\omega}{d\gamma} \tag{5.72}$$

Massive object on weighted string Substitute a massive object of mass M and connected to the ground with spring constant k instead of one of the walls of the coupled masses on the string. Let the displacement of the big mass be z. Let the other end of the string remain connected to a stationary wall. Then the equation of motion of the big mass is:

$$M\ddot{z} + kz = \frac{\tau}{d}(y_1 - z) \tag{5.73}$$

while the other masses still obey the usual equation of motion. Note that in the limit as $d \to 0$:

$$\frac{y_1 - z}{d} \to \frac{\partial y}{\partial x} \tag{5.74}$$

Hence we want to solve the wave equation for the other masses with the boundary condition:

$$y(x = 0) = z(t) \tag{5.75}$$

$$M\ddot{z} + kz = \tau \frac{\partial y}{\partial z} \tag{5.76}$$

$$\frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \tag{5.77}$$

We want the solution as a combination of waves moving left and right:

$$y = y_L + y_R = y_L(x + vt) + y_R(x - vt) \tag{5.78}$$

We hence have the two differential equations (to check how to obtain):

$$\frac{1}{v}\frac{\partial y_L}{\partial t} - \frac{\partial y_L}{\partial x} = 0 \tag{5.79}$$

$$\frac{1}{v}\frac{\partial y_R}{\partial t} + \frac{\partial y_R}{\partial x} = 0 \tag{5.80}$$

Now we are only interested in wave propagating to the right (do not want to drive the mass with left-moving waves). Then we have that (noting that the string is connected to the mass and hence must have the same velocity at $x = 0$):

$$\frac{\partial y}{\partial x} = -\frac{1}{v}\frac{\partial y}{\partial t} = -\frac{1}{v}\frac{dz}{dt} \tag{5.81}$$

$$\implies M\ddot{z} + kz = -\frac{\tau}{v}\frac{dz}{dt} \tag{5.82}$$

$$\implies M\ddot{z} + \frac{\tau}{v}\dot{z} + kz = 0 \tag{5.83}$$

and we observe that the big mass is damped by the string. This is one way to model friction and the flow of energy away from the mass.

Chapter 6

Week 6

6.1 Tuesday, 3 Nov 2015

6.1.1 1D Motion

1D Lagrangian We can always rescale the coordinates to write the Lagrangian as:

$$L = \frac{\dot{q}^2}{2} - V(q) \tag{6.1}$$

and the Hamiltonian will be:

$$E = \frac{\dot{q}^2}{2} + V(q) \tag{6.2}$$

Example: 1D Pendulum with unit length and unit mass.

$$E = \frac{\dot{\theta}^2}{2} + (1 - \cos\theta) = E \tag{6.3}$$

Phase space Plot $\dot{\theta}$ against θ. The trajectory must have constant E. Hence the phase space trajectories are level sets of E. For small energies, we approximate $\cos\theta$ by the quadratic approximation, which gives $\frac{\dot{\theta}^2}{2} + \frac{\theta^2}{2} = E$, which are circles around the origin.

```
ContourPlot[y^2/2 + (1 - Cos[x]), {x, -2 Pi, 2 Pi}, {y, -2, 2}, AspectRatio → 1/Pi, FrameLabel → {"θ", "θ Dot"}]
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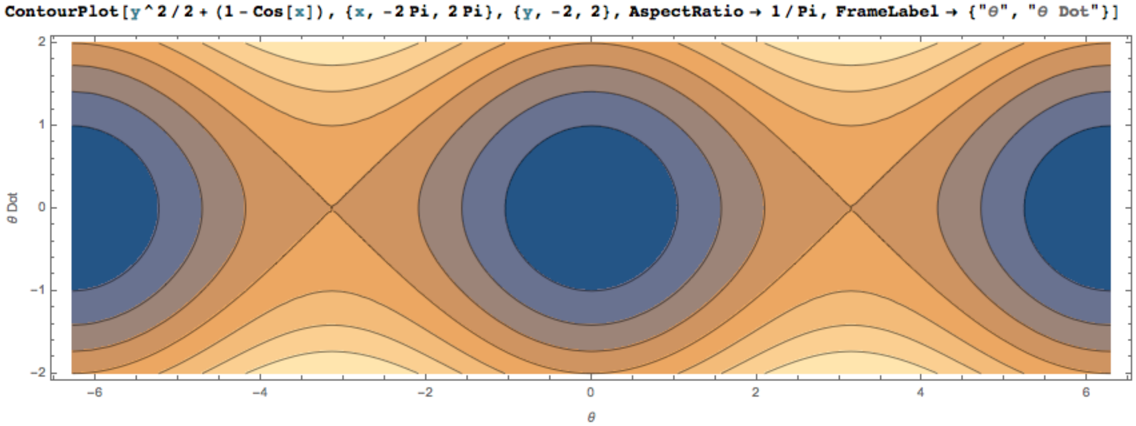

Note that when the energy reaches a critical value, there is a crossing point at $\theta = \pm\pi$ that the system does not actually achieve in finite time. The crossing point is called a separatrix, which separates the phase space into distinct regions (bound and unbound motion).

Consider a pendulum that is bound so that it achieves a maximum angle θ_{max}. Then by energy conservation:

$$\frac{\dot{\theta}^2}{2} + (1 - \cos\theta) = 1 - \cos\theta_{max} \tag{6.4}$$

$$\implies \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_{max})}} = dt \tag{6.5}$$

$$\implies \frac{d\theta}{\sqrt{\sin^2\frac{\theta_{max}}{2} - \sin^2\frac{\theta}{2}}} = 2dt \tag{6.6}$$

Re-parametrizing, define α so that:

$$\sin\frac{\theta}{2} = \sin\frac{\theta_{max}}{2}\sin\alpha \tag{6.7}$$

since we expect that $\theta \leq \theta_{max}$. The integral becomes:

$$T = 4\int_0^{\pi/2} \frac{1}{\sqrt{1 - (\sin^2\frac{\theta_{max}}{2})\sin^2\alpha}} d\alpha \tag{6.8}$$

This is an elliptic integral. The elliptic integral of the first type is:

$$F(\phi|m) = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2\theta}} d\theta \tag{6.9}$$

If $\phi = \pi/2$, then the integral is called a complete elliptic integral of the first type.

For small θ_{max}, we can Taylor expand the integrand to get an approximation:

$$T = 4\int_0^{\pi/2}\left[1 + \frac{1}{2}\sin^2\frac{\theta_{max}^2}{2}\sin^2\alpha\right]d\alpha \approx 4\int_0^{\pi/2}\left[1 + \frac{1}{2}\frac{\theta_{max}^2}{4}\sin^2\alpha\right]d\alpha = 2\pi\left(1 + \frac{\pi\theta_{max}^2}{16} + \cdot\right) \tag{6.10}$$

where we note that the leading order term is the first order approximation if we had made the expansion $\sin\theta \approx \theta$. The second order correction term is positive, implying that the period increases as θ_{max} increases, which is consistent with intuition since the cosine potential falls off more rapidly as θ increases than the quadratic potential approximation.

6.1.2 Central Force

3D motion to 2D motion First consider the full Lagrangian:

$$L = \frac{M_1(\dot{\vec{r}}_1)^2 + M_2(\dot{\vec{r}}_2)^2}{2} - V(|\vec{r}_1 - \vec{r}_2|) \tag{6.11}$$

Note that the system exhibits displacement symmetry. That is, if we displace the coordinates by a fixed constant, the Lagrangian remains the same:

$$\tilde{L}(\vec{r}_j, \dot{\vec{r}}_j, \vec{a}) = L(\vec{r}_1 + \vec{a}, \vec{r}_2 + \vec{a}, \dot{\vec{r}}_1, \dot{\vec{r}}_2) \implies \frac{\partial\tilde{L}}{\partial\vec{a}} = 0 \tag{6.12}$$

$$\implies \frac{\partial L}{\partial\vec{r}_1} + \frac{\partial L}{\partial\vec{r}_2} = 0 \tag{6.13}$$

$$\implies \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{6.14}$$

Hence there is an ignorable coordinate. We define the center of mass coordinate:

$$\vec{R} = \frac{M_1\vec{r}_1 + M_2\vec{r}_2}{M_1 + M_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \tag{6.15}$$

and the Lagrangian becomes:

$$L = \frac{M}{2}(\dot{\vec{R}})^2 + \frac{\mu}{2}(\dot{\vec{r}})^2 - V(r) \tag{6.16}$$

Clearly, R is an ignorable coordinate. Proceed by use of a Routhian. Then we just need to solve for a one-body motion in terms of the coordinate r.

Proof that motion is in a plane Move into spherical coordinates. Then the Lagrangian for the CM frame (removing the constant effect of the CM motion) is:

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta + \dot{\phi}^2\right) - V(r) \tag{6.17}$$

Clearly, ϕ is ignorable, hence p_ϕ is a constant. But this means that:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2\theta\dot{\phi} \tag{6.18}$$

is conserved. But this is precisely the angular momentum with respect to the z-axis. Note that the angular momentum can be written as:

$$\vec{L}_{AM} = \vec{r} \times (\mu\dot{\vec{r}}) \tag{6.19}$$

and since it is a constant, both $\vec{r}$ and $\dot{\vec{r}}$ must lie in the plane orthogonal to $\vec{L}_{AM}$. Moving into this plane, we realize that θ can just be taken to be $\frac{\pi}{2}$ so that the Lagrangian becomes:

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r) \tag{6.20}$$

The total energy is given by:

$$E = \frac{\mu\dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + V(r) = \text{constant} \tag{6.21}$$

Hence the strategy is to obtain $r(t)$ by solving the 1D motion, then obtaining $\dot{\phi} = \frac{l}{\mu r(t)^2}$, and integrating to obtain $\phi(t)$ to fully describe the system.

6.1.3 Keplerian central force motion

Consider $V = \frac{-k}{r}$. Conservation of energy gives:

$$E = \frac{\mu\dot{r}^2}{2} + \frac{l^2}{2\mu r^2} - \frac{k}{r} \tag{6.22}$$

Consider the phase space plot of this system, $\dot{r}$ against r.

```
ContourPlot[y^2 + .1 / x^2 - 1 / x, {x, 0, 1}, {y, -2, 2}, AspectRatio → 0.5, FrameLabel → {"r", "r Dot"}]
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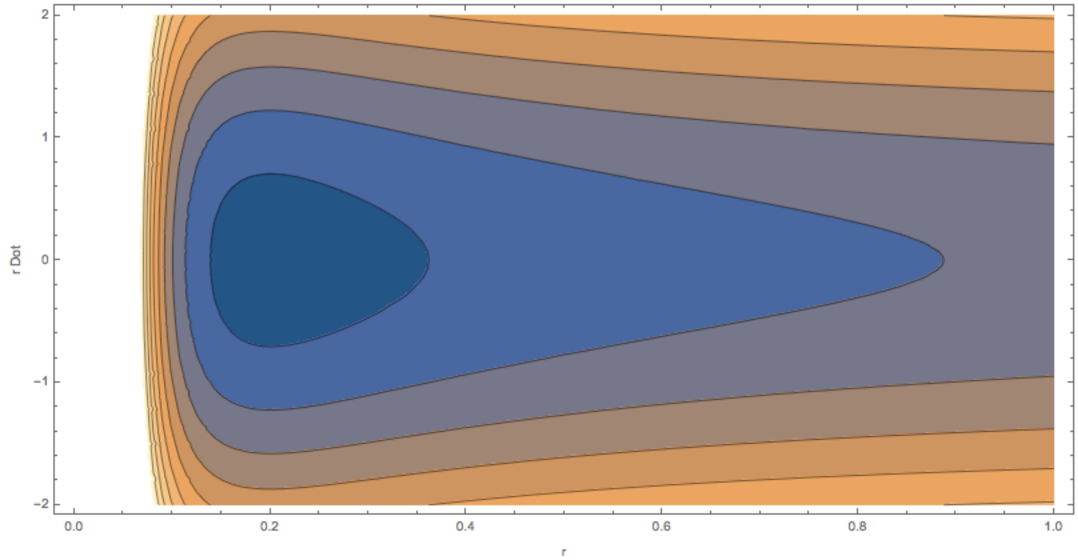

The separatrix corresponds to the level set that first extends to infinity.

The equation of motion in 1D can be re-written as:

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)} \tag{6.23}$$

$$\dot{\phi} = \frac{l}{\mu r^2} \tag{6.24}$$

and taking the ratio of these two, we obtain:

$$\frac{\dot{\phi}}{\dot{r}} = \frac{d\phi}{dr} = \frac{l}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}} \tag{6.25}$$

Now further that if we define $u = \frac{1}{r}$, the energy of the system can be written as:

$$\frac{l^2}{2\mu}\left(\frac{du}{d\phi}\right)^2 + \frac{l^2}{2\mu}u^2 - ku = E \tag{6.26}$$

which is a harmonic oscillator conservation of energy equation (somewhat)! Taking the derivative of this equation with respect to ϕ,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{l^2} \tag{6.27}$$

$$\implies u = \frac{1}{r} = \frac{\mu k}{l^2} + A\cos\phi, \quad A \in \mathbb{R} \tag{6.28}$$

Define $p = \frac{l^2}{\mu k}$, which has a unit of distance. Also define $pA = \epsilon$. Then we have:

$$r = \frac{P}{1 + \epsilon\cos\phi} \tag{6.29}$$

which is the equation of a conic section. If $\epsilon = 0$, it is a circle, if $\epsilon < 1$, we have an ellipse, if $\epsilon = 1$ we have a parabola, if $\epsilon > 1$ we have a hyperbola.

The energy of the system can be parametrized in terms of ϵ:

$$E = -\frac{k}{2P}(1 - \epsilon^2) \tag{6.30}$$

6.2 Thursday 5 Nov 2015

Keplerian motion: Elliptic case Recall the parametrization of the ellipse:

$$r = \frac{P}{1 + \epsilon \cos \phi} \tag{6.31}$$

with parameters:

$$P = \frac{l^2}{\mu k} \tag{6.32}$$
$$E = -\frac{k}{2P}(1 - \epsilon^2) \tag{6.33}$$

The periapsis and apoapsis are given by:

$$r_{peri} = \frac{P}{1 + \epsilon} \tag{6.34}$$
$$r_{apo} = \frac{P}{1 - \epsilon} \tag{6.35}$$

and the semi-major and semi-minor axes are:

$$a = \frac{P}{1 - \epsilon^2} \tag{6.36}$$
$$b = \frac{P}{\sqrt{1 - \epsilon^2}} \tag{6.37}$$

The focal length c is the distance between the focus and the origin, and is given by:

$$c = \frac{\epsilon P}{1 - \epsilon^2} \tag{6.38}$$

The relation between a, b and c is:

$$b^2 + c^2 = a^2 \tag{6.39}$$

which can be proven by noting that the sum of the distances of any point on the ellipse to the two foci is equal and will have value $2a$.

Note that the area swept up by the radial vector is:

$$dA = \frac{1}{2} r^2 d\phi \implies \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{l}{2\mu} \tag{6.40}$$

Hence angular momentum conservation implies Kepler's second law. We can also use this to calculate the period of the orbit by knowing the total area of the ellipse:

$$\pi ab = T \frac{l}{2\mu} \implies T = 2\pi \sqrt{\frac{\mu a^3}{k}} \tag{6.41}$$

Explicit time dependence We parametrize the ellipse:

$$X = a \cos \psi \tag{6.42}$$
$$Y = b \sin \psi \tag{6.43}$$

where ψ is measured from the origin. The angle ϕ can be defined similarly with reference to the focus. By convention, ϕ is called the true anomaly and ψ is called the eccentric anomaly.

The radial coordinate can be related to ψ using:

$$r = a(1 - \epsilon \cos\psi) \tag{6.44}$$

We note that we can relate the time and the radial coordinate by rearranging the conservation of energy equation:

$$t = \sqrt{\frac{\mu}{2}} \int \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{l^2}{2\mu r^2}}} \tag{6.45}$$

and inserting the expression for $r(\psi)$:

$$t = \sqrt{\frac{\mu a^3}{k}} \int d\psi (1 - \epsilon \cos\psi) = \sqrt{\frac{\mu a^3}{k}} (\psi - \epsilon \sin\psi) \tag{6.46}$$

This implicitly defines $\psi(t)$. Clearly, as $\psi \to \psi + 2\pi$, we re-obtain the expression for the period: $T = 2\pi\sqrt{\frac{\mu a^3}{k}}$.

Hyperbolic Motion We parametrize the left branch of the hyperbola:

$$X = -a \cosh\psi \tag{6.47}$$
$$Y = b \sinh\psi \tag{6.48}$$

where ψ is measured from the origin. The corresponding implicit definition of $\psi(t)$ is given by:

$$T(\psi) = \sqrt{\frac{\mu a^3}{k}} (\epsilon \sinh\psi - \psi) \tag{6.49}$$

Scattering problem Consider a repulsive central force problem $k < 0$. We then define:

$$P = -\frac{l^2}{\mu k} \tag{6.50}$$

which gives the polar representation of the orbit:

$$r = \frac{P}{\epsilon \cos\phi - 1} \tag{6.51}$$

where ϕ is measured from the focus. This corresponds to the other branch of the hyperbola that turns away from the focus.

Closed Orbits Define $\Delta\phi$ to be the angular difference between two instances where $r = r_{max}$ that is separated by one instance of $r = r_{min}$. Note that in the Keplerian motion, $\Delta\phi = 2\pi$ so that the orbit is closed. We can consider a perturbation to the Kepler potential so that $\Delta\phi = 2\pi + \delta$, where δ is small. Hence for each orbit, the perihelion precesses by δ.

Potential and closed orbits Now we want the condition on the potential so that the orbit is closed. Note that the condition for closed orbits is for Δ to be a rational multiple of π:

$$\Delta\phi = \frac{m}{n}\pi, \quad m, n \in \mathbb{Z} \tag{6.52}$$

Note that $\Delta\phi$ corresponds to the radial motion undergoing one periodic cycle. We want to find the period of the angular motion.

Consider a perturbation on r. Define the equilibrium radius r_0 that satisfies:

$$-\frac{l^2}{\mu r_0^3} + V'(r_0) = 0 \tag{6.53}$$

Then expand $E(r)$ around r_0. The first derivative must vanish because r_0 is an equilibrium point.

$$\frac{\dot{r}^2}{2} + \frac{1}{2}\left[\frac{3l^2}{\mu^2 r_0^4} + V''(r_0)\right](r - r_0)^2 = E \tag{6.54}$$

This gives a Harmonic oscillator potential which has an angular frequency given by:

$$\Omega = \sqrt{\frac{3l^2}{\mu^2 r_0^4} + V''(r_0)} = \frac{l}{\mu r_0^2}\sqrt{3 + \frac{r_0 V''}{V'}} \tag{6.55}$$

so that the radial motion is given by:

$$r = r_0 + \delta \cos \Omega t \tag{6.56}$$

The angular motion is given by the angular momentum conservation equation:

$$\dot{\phi} = \frac{l}{\mu r^2} = \frac{l}{\mu r_0^2 (1 + \frac{\delta}{r_0}\cos \Omega t)^2} \approx \frac{l}{\mu r_0^2}\left(1 - \frac{2\delta \cos \Omega t}{r_0}\right) \tag{6.57}$$

Integrating over 1 cycle of radial motion $t : 0 \to \frac{2\pi}{\Omega}$, we obtain that:

$$\Delta\phi = \frac{l}{\mu r_0^2}\frac{2\pi}{\Omega} + O(\delta^2) \tag{6.58}$$

Considering the first order terms:

$$\Delta\phi = \frac{2\pi}{\sqrt{3 + \frac{r\frac{dV'}{dr}}{V'}}} = \frac{2\pi}{\sqrt{3 + \frac{d\log V'}{d\log r}}} \tag{6.59}$$

Hence for $\Delta\phi$ to be a rational multiple of π at all times and all positions, we require that $\frac{d\log V'}{d\log r}$ be a constant. But this means that V has to be a power-law function of r. Then we have the necessary condition for closed orbits:

$$V = \alpha r^n \tag{6.60}$$

To obtain the sufficient condition, we need to ensure that $\sqrt{3 + (n-1)} = \sqrt{2+n}$ is rational. Clearly, $n = -1$ for the Keplerian motion works. Also, the Harmonic potential $n = 2$ works.

6.2.1 Scattering Problem

Let particles with energy E move in the x direction towards a target particle at the origin. Depending on the distance of the particles from the x-axis, the scattering effect will be different. Define the impact parameter s to be the vertical distance between the horizontal incoming particle path and the x-axis. The impact parameter is then related to the scattering angle.

Note that the solid angle around the target particle can be written in terms of the angle made with the scattered particles:

$$d\Omega = 2\pi \sin\theta d\theta \tag{6.61}$$

The particles that pass through this solid angle can be traced back to a ring of incoming particles with impact parameters:

$$dN = R 2\pi s ds \tag{6.62}$$

where R is the incoming flux. Then we have the equality:

$$\frac{1}{R}\frac{dN}{d\Omega} = \frac{s ds}{\sin\theta d\theta} \tag{6.63}$$

This gives us the differential cross section on the RHS. This cross section gives us the relation between θ and s. Hence by measuring the number of particles per unit solid angle and divided by the incident rate, we can calculate the differential cross section.

For the Keplerian problem, the relation between the impact parameter and the exit angle is:

$$s = \frac{k}{2E}\cot\frac{\theta}{2} \tag{6.64}$$

so that the differential cross section is:

$$\sigma(\theta) = \frac{k^2}{16E^2}\frac{1}{\sin^4(\theta/2)} \tag{6.65}$$

6.2.2 Virial theorem

Define:

$$G = \sum_j \vec{p}_j \cdot \vec{r}_j \tag{6.66}$$

Calculating the total time derivative:

$$\frac{dG}{dt} = \sum_j \vec{F}_j \cdot \vec{r}_j + \sum_j \vec{p}_j \cdot \dot{\vec{r}}_j = \sum_j \vec{F}_j \cdot \vec{r}_j + 2T_{tot} \tag{6.67}$$

If the potential has the form:

$$V_{tot} = \frac{1}{2}\sum_{j\neq k} V(|\vec{r}_j - \vec{r}_k|) \tag{6.68}$$

where the individual potentials are power law types $V = kr^n$, then:

$$\sum_j \vec{F}_j \cdot \vec{r}_j = -nV_{tot} \tag{6.69}$$

Hence:

$$2T_{tot} - nV_{tot} = \frac{dG}{dt} \tag{6.70}$$

For a system in steady state, we can take the average of $\frac{dG}{dt}$ over time, which should vanish. Then the time average of the kinetic and potential energies give:

$$2\bar{T}_{tot} - n\bar{V}_{tot} = 0 \tag{6.71}$$

Chapter 7

Week 7

7.1 Tuesday, 10 Nov 2015

7.1.1 Noether's Theorem

Example: Azimuthal symmetry Consider a frame that is rotated about the z axis at an angle θ to the x-axis. We know that the Lagrangian will be invariant under this coordinate transform. We proceed in the Cartesian coordinate system, pretending that we are unable to construct a coordinate system with the ignorable ϕ coordinate.

To have a symmetry, we require that the functional form (they are the same function!) of the Lagrangian be identical in both the transformed and untransformed coordinates:

$$L'(x', y', z', \dot{x}', \dot{y}', \dot{z}', t) = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

Hence L must be independent of the parameter θ. Hence we can take the partial derivative of $L(x', y', \dots)$ with respect to θ around $\theta = 0$ and require that it be zero. We will get that (along the trajectory so that we can apply the Euler-Lagrange equations):

$$\frac{d}{dt}\left[p_x y - p_y x\right] = 0$$

and hence the angular momentum in the z-direction is zero.

General proof of Noether's Theorem Suppose we have a set of coordinates q_k, where $k = 1, 2, \dots, N$. Suppose we have a map parametrized by s such that:

$$q_k \to Q_l(q_k, s) \tag{7.1}$$
$$Q_l(q_k, 0) = q_l \tag{7.2}$$

Define the derivative:

$$\gamma_l(Q_k) = \left.\frac{\partial Q_l}{\partial s}\right|_{(q_k, s)} \tag{7.3}$$

Note that γ_l does not depend on s! That is, the transformation is linear in s. Any infinitesimal transformation can be written in terms of a first order linear transformation.

A symmetry of the system means that:

$$L(Q_l(q_k, s), \dot{Q}_l(q_k, \dot{q}_k, s), t) \tag{7.4}$$

is independent of s. Note further that:

$$\dot{Q}_l = \sum_k \frac{\partial Q_l}{\partial q_k} \dot{q}_k \tag{7.5}$$

Taking the derivative of the Lagrangian with respect to s,

$$\frac{\partial L}{\partial s} = \sum_k \frac{\partial L}{\partial q_k} \frac{\partial Q_k}{\partial s} + \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{Q}_k}{\partial s} = 0 \tag{7.6}$$

But by the equality of mixed partial derivatives:

$$\frac{\partial \dot{Q}_k}{\partial s} = \sum_m \left[\frac{\partial}{\partial q_m} \frac{\partial Q_k}{\partial s} \right] \dot{q}_m = \frac{d}{dt} \frac{\partial Q_k}{\partial s} \tag{7.7}$$

Hence we may combine this with the derivative of the Lagrangian to get:

$$\frac{d}{dt} \left[\sum_k p_k \frac{\partial Q_k}{\partial s} \right] = 0, \quad p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{7.8}$$

$$\implies \sum_k p_k \gamma_k = \text{ constant} \tag{7.9}$$

Back to rotations Note that the transformed coordinate vector can be written as:

$$\vec{X} = \vec{x} + s\vec{n} \times \vec{x} \tag{7.10}$$

where $\vec{n}$ is a constant vector that we are rotating around. In terms of components:

$$X_j = x_j + s \sum_{k,l} \epsilon_{jkl} n_k x_l \tag{7.11}$$

so that the components of the gamma vector are:

$$\gamma_j = \sum_{k,l} \epsilon_{jkl} n_k x_l \tag{7.12}$$

The cross product was written in terms of the Levi-Civita symbol:

$$(\vec{A} \times \vec{B})_j = \sum_{k,l} \epsilon_{jkl} A_k B_l \tag{7.13}$$

Then the constant of motion is:

$$I = \sum_j p_j \gamma_j \tag{7.14}$$

$$= \sum_{j,k,l} p_j \epsilon_{jkl} n_k x_l \tag{7.15}$$

$$= \sum_k n_k \sum_{j,l} \epsilon_{jkl} x_l p_j \tag{7.16}$$

$$= \sum_k n_k \sum_{j,l} \epsilon_{klj} x_l p_j \tag{7.17}$$

$$= \vec{n} \cdot (\vec{x} \times \vec{p}) \tag{7.18}$$

and we recognize this to be the $\hat{n}$ component of the angular momentum. Hence the invariance of the Lagrangian under a rotation about $\vec{n}$ corresponds to the conservation of the momentum about that axis.

Actually the easy way to do this without the Levi-Civita symbol is to use the vector triple product:

$$I = \vec{p} \cdot \vec{\gamma} = \vec{p} \cdot (\vec{n} \times \vec{x}) = \vec{n} \cdot (\vec{x} \times \vec{p}) \tag{7.19}$$

7.1.2 Legendre Transform

Recall from Thermodynamics Recall that the first law of thermodynamics can be written as:

$$dU(S,V) = TdS - PdV \tag{7.20}$$

where the parameters are related by:

$$\left(\frac{\partial U}{\partial S}\right)_V = T, \quad \left(\frac{\partial U}{\partial V}\right)_S = -P \tag{7.21}$$

If we want to use T and V to parametrize the system, we use the Helmholtz free energy $F = U - TS$, the Legendre transform of the internal energy U, so that $dF = -SdT - pdV$ is written in differentials of T and V.

Mathematical definition Suppose we have parameters y, z and two functions $A(y), B(z)$. Suppose that:

$$z = \frac{dA(y)}{dy} \tag{7.22}$$

Then define $B(z) = yz - A(y)$ to be the Legendre transform of $A(y)$. Note that in the definition of $B(z)$, the values of y are obtained from z by inverting $z = \frac{dA(y)}{dy}$.

The total derivative of $B(z)$ is:

$$dB = ydz + zdy - dA = ydz + zdy - \frac{dA}{dy}dy = ydz + dy\left(z - \frac{dA}{dy}\right) = ydz \tag{7.23}$$

$$\implies \frac{dB(z)}{dz} = y \tag{7.24}$$

The following properties hold:

$$A + B = yz \tag{7.25}$$

$$dA = zdy \tag{7.26}$$

$$dB = ydz \tag{7.27}$$

$$\implies dA + dB = zdy + ydz \tag{7.28}$$

In order to solve for the transform in a unique fashion, we require that A is convex:

$$\frac{d^2A}{dy^2} > 0 \tag{7.29}$$

The geometric interpretation is that if $A(y)$ is plotted against y, then the slope is given by z, and the linear approximation at a point has a vertical axis intercept given by $-B$.

Relation to Lagrangian and Hamiltonian dynamics Recall that:

$$p = \frac{\partial L}{\partial \dot{q}} \tag{7.30}$$

so $z \leftrightarrow p, y \leftrightarrow \dot{q}$. Hence the Legendre transformation of the Lagrangian is:

$$p\dot{q} - L = H \tag{7.31}$$

so that the partial derivative of H with respect to p and q will be simple. Note that $\dot{q}$ is represented by inverting the definition of the canonical momentum so that it is expressed in terms of p, q.

Derivatives of the Hamiltonian

$$dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d(\dot{q}) - \frac{\partial L}{\partial t}dt = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{7.32}$$

$$\implies \frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\dot{p} \tag{7.33}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{7.34}$$

Total derivative of the Hamiltonian

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{7.35}$$

Hence the total time derivative of the Hamiltonian is equal to the partial time derivative as well! Hence:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{7.36}$$

7.2 Thursday, 12 Nov 2015

Review of Noether's Theroem Consider a continuous map parametrized by s: $Q_k(q_l, s)$ so that $Q_k(q_l, 0) = q_k$ and $\frac{\partial Q_k}{\partial s} = \gamma_k(Q_l)$, where γ_k is independent of s. Defining the canonical momenta:

$$p_k = \left.\frac{\partial L}{\partial \dot{q}_k}\right|_{Q_l, \dot{Q}_l, t} \tag{7.37}$$

The following quantity is then conserved in time:

$$I = \sum_k p_k \gamma_k \tag{7.38}$$

Example 1: SHO Consider a Lagrangian:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{7.39}$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{7.40}$$

$$\implies \dot{q} = \frac{p}{m} \tag{7.41}$$

$$H = p\dot{q} - L = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{7.42}$$

The Hamilton equations of motion are hence:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{7.43}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq \tag{7.44}$$

Example 2: Coupled oscillators

$$L = \frac{1}{2}\sum_{k,l}\dot{q}_k t_{kl}\dot{q}_l - \frac{1}{2}\sum_{k,l} q_k v_{kl} q_l = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{t}\dot{\boldsymbol{q}} - \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} \tag{7.45}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{t}\dot{\boldsymbol{q}} \tag{7.46}$$

$$H = \boldsymbol{p}\cdot\dot{\boldsymbol{q}} - L = \boldsymbol{p}^T\dot{\boldsymbol{q}} - L = (\dot{\boldsymbol{q}}^T\boldsymbol{t})\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{t}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{t}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} \tag{7.47}$$

$$\implies H = \frac{1}{2}(\boldsymbol{t}^{-1}\boldsymbol{p})^T\boldsymbol{t}(\boldsymbol{t}^{-1}\boldsymbol{p}) + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} = \frac{1}{2}\boldsymbol{p}^T(\boldsymbol{t}^{-1})^T\boldsymbol{t}\boldsymbol{t}^{-1}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} \tag{7.48}$$

$$\implies H = \frac{1}{2}\boldsymbol{p}^T(\boldsymbol{t}^{-1})^T\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{t}^{-1}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{v}\boldsymbol{q} \tag{7.49}$$

where we note that $\boldsymbol{t}^{-1} = (\boldsymbol{t}^{-1})^T$ because $\boldsymbol{t}$ and $\boldsymbol{t}^{-1}$ are both symmetric. The canonical equations are:

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} = \boldsymbol{t}^{-1}\boldsymbol{p} \tag{7.50}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} = -\boldsymbol{v}\boldsymbol{q} \tag{7.51}$$

Example 3: Spherical coordinates

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \tag{7.52}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{7.53}$$

$$p_\theta = mr^2\dot{\theta} \tag{7.54}$$

$$p_\phi = mr^2\sin^2\theta\dot{\phi} \tag{7.55}$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \tag{7.56}$$

$$\implies H = \frac{p_r^2}{2m} + \frac{m}{2}r^2\left(\frac{p_\theta}{mr^2}\right)^2 + \frac{m}{2}r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2 = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} \tag{7.57}$$

The sum of the second and third terms is the total angular momentum squared.

Liouville's Theorem Consider a phase space density $\rho(p, q, t)$ corresponding to the number of particle trajectories per unit phase space. Draw a box from $(q, q + \Delta q) \times (p, p + \Delta p)$. We want to find the rate of change of the number of particles contained in that box. In a small time interval Δt:

$$\Delta N = \int_p^{p+\Delta p} dp' \Delta t \dot{q}(q, p', t)\rho(q, p', t) - \int_p^{p+\Delta p} dp' \Delta t \dot{q}(q + \Delta q, p', t)\rho(q + \Delta q, p', t) \tag{7.58}$$

$$+ \int_q^{q+\Delta q} dq' \Delta t \dot{p}(q', p, t)\rho(q', p, t) - \int_q^{q+\Delta t} dq' \Delta t \dot{p}(q', p + \Delta p, t)\rho(q', p + \Delta p, t) \tag{7.59}$$

$$\implies \Delta N \approx \left[-\frac{\partial}{\partial q}(\dot{q}\rho)\right]\Delta q \Delta p \Delta t + \left[-\frac{\partial}{\partial p}(\dot{p}\rho)\right]\Delta q \Delta p \Delta t \tag{7.60}$$

$$\implies \frac{\Delta \rho}{\Delta t} = -\frac{\partial}{\partial q}(\dot{q}\rho) - \frac{\partial}{\partial p}(\dot{p}\rho) \tag{7.61}$$

In the limit where $\Delta t \to 0$:

$$\frac{\partial \rho}{\partial t} = -\rho\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right) - \dot{q}\frac{\partial \rho}{\partial q} - \dot{p}\frac{\partial \rho}{\partial p} \tag{7.62}$$

This is a continuity equation. Consider the fluid analogue:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \implies \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{7.63}$$

$$\implies \frac{\partial \rho}{\partial t} + \sum_j v_j \frac{\partial \rho}{\partial x_j} + \rho(\nabla \cdot \vec{v}) = 0 \tag{7.64}$$

The term $\frac{\partial \rho}{\partial t} + \sum_j v_j \frac{\partial \rho}{\partial x_j}$ refers to the change in ρ following a particle trajectory. Write it as a total derivative. Then:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \tag{7.65}$$

Similarly, we express the phase space conservation law accordingly:

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right) = 0 \tag{7.66}$$

We can generalize this result to a large number of dimensions. Then:

$$\frac{d\rho}{dt} + \rho \sum_k \left(\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} \right) = 0 \tag{7.67}$$

Using the canonical equations of Hamiltonian dynamics:

$$\frac{d\rho}{dt} + \rho \sum_k \left(\frac{\partial \frac{\partial H}{\partial p_k}}{\partial q_k} + \frac{\partial \frac{-\partial H}{\partial q_k}}{\partial p_k} \right) = 0 \tag{7.68}$$

$$\implies \frac{d\rho}{dt} + \rho \sum_k \left(\frac{\partial^2 H}{\partial q_k \partial p_k} - \frac{\partial^2 H}{\partial p_k \partial q_k} \right) = 0 \tag{7.69}$$

$$\implies \frac{d\rho}{dt} = 0 \tag{7.70}$$

by the equality of mixed particle derivatives.

Consider an initial condition where all the particle trajectories are initially contained in a volume Ω, so that:

$$\rho = \begin{cases} \rho_0, & \text{inside } \Omega \\ 0, & \text{outside} \end{cases} \tag{7.71}$$

by the conservation of particle number:

$$\int \rho dp dq = N = \rho_0 V(\Omega) \tag{7.72}$$

Now let the system evolve. Let the trajectories be contained in a new volume Ω' after time t. By the statement that $\frac{d\rho}{dt} = 0$ earlier, we know that the density of phase space cannot change along the trajectories. Then:

$$\rho(t) = \begin{cases} \rho_0, & \text{inside } \Omega' \\ 0, & \text{outside} \end{cases} \tag{7.73}$$

Implementing the conservation of particle number,

$$\int \rho dp dq = N = \rho_0 V(\Omega') \tag{7.74}$$

But this means that $V(\Omega') = V(\Omega)$. Hence the volume occupied by the particle trajectories does not change in time.

Poincare Recurrence Theorem Consider a system bound in some finite region of phase space. Consider a Hamiltonian that is independent of time. Weak version: For any point of phase space $\mathcal{P}$, in any neighbourhood U of $\mathcal{P}$, all points of U either stay in U forever, or there exists 1 point $x \in U$ that leaves U and then returns.

Proof of weak form Suppose a point Q leaves U and reaches R. Pick a region around Q called Ω_0 such that $\Omega_0 \in U$. Then Ω_0 evolves to $\Omega_1 \notin U$ by making Ω_0 small enough. We can repeat this analysis to get a sequence of regions $\{\Omega_i\}$. But since the phase space volume is finite, there will exist $k < n$ such that $\Omega_k \cap \Omega_n \neq 0$, and there is some overlap between the regions occupied by the phase volume at two times. But this also means that Ω_{k-1} and Ω_{n-1} must have overlapped as well, because they have the same volume at all times. We can trace this backwards to declare that Ω_0 must have overlapped with Ω_{n-k}.

Formulation of action in the Hamiltonian picture

$$S[p,q] = \int_{t_1}^{t_2} (p\dot{q} - H) dt \tag{7.75}$$

$$\implies \delta S[p,q] = \int_{t_1}^{t_2} \left(\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q \right) dt \tag{7.76}$$

$$= \int_{t_1}^{t_2} \left(\dot{q}\delta p + \frac{d}{dt}(p\delta q) - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q \right) dt \tag{7.77}$$

where we only consider trajectories that start at a fixed q_1 and end at a fixed q_2. That is, $\delta q(t_1) = \delta(t_2) = 0$. Rearranging,

$$\int_{t_1}^{t_2} \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] dt = 0 \tag{7.78}$$

which simply gives us the canonical Hamilton equations.

Chapter 8

Week 8

8.1 Tuesday 17 Nov 2015

8.1.1 Canonical transformations

Coordinate transforms in Lagrangian dynamics Recall that we can write $Q(q,t)$ so that the velocity $\dot{Q}(q,\dot{q},t) = \frac{\partial Q(q,t)}{\partial q}\dot{q} + \frac{\partial Q(q,t)}{\partial t}$ is given automatically. Then the new Lagrangian $\bar{L}$ must satisfy:

$$\bar{L}(Q(q,t),\dot{Q}(q,\dot{q},t),t) = L(q,\dot{q},t) \tag{8.1}$$

Canonical transformation We want to find the maps $Q(q,p,t)$ and $P(q,p,t)$ that satisfy Hamilton's equations. Note that defining Q does not automatically define P, unlike the case of the point transformation $Q(q,t)$ earlier (since $\dot{Q}$ is automatically defined from Q).

Constraints on the canonical transformation The basis of the constraint is Liouville's theorem (volume of phase space remains constant). The area of the image of the phase space volume under the transformations P, Q must remain constant. This can be written as the Jacobian:

$$\frac{\partial(P,Q)}{\partial(p,q)} = \text{ constant}$$

The constant turns out to be $\frac{A'}{A}$, the ratio of the volume under P, Q to the volume without the map. We hence write the condition for a canonical transformation:

$$\det\begin{pmatrix} \left(\frac{\partial Q}{\partial q}\right)_{p,t} & \left(\frac{\partial Q}{\partial p}\right)_{q,t} \\ \left(\frac{\partial P}{\partial q}\right)_{p,t} & \left(\frac{\partial P}{\partial p}\right)_{q,t} \end{pmatrix} = C$$

Suppose we want the area under the mapping to be the same. That is, $C = 1$. This can always be done by rescaling P, Q and H so that the determinant is ± 1. More precisely, the rescaling will take $C \to \lambda^2 C$. Switching $P \leftrightarrow Q$ will change the sign of the constant $C \to -C$. We can just choose the constant to be $+1$ by convention.

Poisson Bracket Define:

$$\{A(q,p,t), B(q,p,t)\}_{q,p} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

By comparison to the condition on the Jacobian determinant, we have the necessary condition of the canonical transformation:

$$\{Q,P\}_{q,p} = 1 \tag{8.2}$$

Generating Function First construct a class of maps from $q, p \to Q, P$. We want to find $\tilde{H}$ so that:

$$\delta \int_{t_1}^{t_2} \left[P\dot{Q} - \tilde{H}(Q,P,t)\right] dt \tag{8.3}$$

is stationary if:

$$\delta \int_{t_1}^{t_2} \left[p\dot{q} - H(q,p,t)\right] dt = 0 \tag{8.4}$$

where we have taken $\delta Q(t_1) = \delta Q(t_2) = 0$.

We can write this condition as:

$$P\dot{Q} - \tilde{H}(Q,P,t) = p\dot{q} - H(q,p,t) - \frac{dF(q,Q,t)}{dt} \tag{8.5}$$

$$\implies P\dot{Q} - \tilde{H}(Q,P,t) = p\dot{q} - H(q,p,t) - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \tag{8.6}$$

Note that we disallowed F to depend on $\dot{q}$ and $\dot{Q}$. Comparing coefficients, we note that if construct F such that:

$$P = -\frac{\partial F}{\partial Q} \tag{8.7}$$

$$p = \frac{\partial F}{\partial q} \tag{8.8}$$

then we obtain that:

$$\tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial F(q,Q,t)}{\partial t} \tag{8.9}$$

Note that it is possible to invert the equations for P, p to complete determine P, Q in terms of p, q.

Given the stationary action condition:

$$\delta \int_{t_1}^{t_2} \left[P\dot{Q} - \tilde{H}(Q,P,t)\right] dt = \delta \int_{t_1}^{t_2} \left[p\dot{q} - H(q,p,t)\right] dt - \delta F(q_2, Q_2, t_2) + \delta F(q_1, Q_1, t_1) \tag{8.10}$$

It can be shown that if p, q satisfy the canonical equations under H, then under the Hamiltonian $\tilde{H}(Q,P,t)$, Hamilton's equations for Q, P also hold:

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} \tag{8.11}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} \tag{8.12}$$

Constructing the actual generating function Given the conditions on the generating function:

$$P = -\frac{\partial F}{\partial Q} \tag{8.13}$$

$$p = \frac{\partial F}{\partial q} \tag{8.14}$$

We can define the function F:

$$F(q_0, Q_0) = \int_O^{q_0, Q_0} -P(dQ) + pdq \tag{8.15}$$

where O is a reference point. Note that this integral definition is valid only if the integral is path-independent. This path independence implies that the closed loop integral must vanish:

$$\oint(-PdQ + pdq) = 0 \tag{8.16}$$

But the loop integral is simply the difference in phase space volume in the two maps. Since we required that the phase space volume remain constant under the mapping, the loop integral is zero, and hence $F(q, Q)$ is defined and satisfies the requirements of the generating function.

Examples using the canonical transformation Consider the generating function $F = qQ$. Then:

$$P = -\frac{\partial F}{\partial Q} = -q \tag{8.17}$$

$$p = \frac{\partial F}{\partial q} = Q \tag{8.18}$$

Note that this transformation exchanges the role of position and momentum (up to a sign change).

Harmonic Oscillator using Canonical transformations Consider the Hamiltonian:

$$H = \frac{p^2 + \omega^2 q^2}{2} \tag{8.19}$$

Define the generating function:

$$F(q, Q) = \frac{1}{2}\omega q^2 \cot(2\pi Q) \tag{8.20}$$

$$p = \frac{\partial F}{\partial q} = \omega q \cot(2\pi Q) \tag{8.21}$$

$$P = -\frac{\partial F}{\partial Q} = \frac{\pi\omega q^2}{\sin^2 2\pi Q} \tag{8.22}$$

Solving these two equations simultaneously,

$$p^2 + \omega^2 q^2 = \frac{\omega^2}{\pi^2} P^2 \tag{8.23}$$

Hence the new Hamiltonian is:

$$\tilde{H}(Q, P, t) = \frac{\omega}{2\pi} P \tag{8.24}$$

Implementing the Hamilton's equations,

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \frac{\omega}{2\pi} \implies Q = Q_0 + \frac{\omega}{2\pi} t \tag{8.25}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \implies P = P_0 \tag{8.26}$$

Inverting the equations for P, Q,

$$q = \sqrt{\frac{P}{\pi\omega}} \sin(2\pi Q) \tag{8.27}$$

$$p = \sqrt{\frac{\omega P}{\pi}} \cos(2\pi Q) \tag{8.28}$$

Hence the amplitude of the oscillation is a constant, and the time dependence is contained inside the trigonometric function.

Generating other canonical transformations

Recall that the generating function was previously obtained by:

$$P\dot{Q} - \tilde{H}(Q, P, t) = p\dot{q} - H(q, p, t) - \frac{dF_1(q, Q, t)}{dt} \tag{8.29}$$

We can rearrange the terms to write:

$$-\dot{P}Q - \tilde{H} = p\dot{q} - H - \frac{d}{dt}(F_1 + PQ) \tag{8.30}$$

Define $F_2(q, P, t) = F_1 + PQ$. Then we can pick the derivatives of F_2 to cancel the other terms:

$$\frac{\partial F_2}{\partial q} = p \tag{8.31}$$

$$\frac{\partial F_2}{\partial P} = Q \tag{8.32}$$

$$\implies \tilde{H} = H + \frac{\partial F_2}{\partial t} \tag{8.33}$$

Why introduce more transformations? $F_2(q, P, t)$ is useful because we can choose transformations that have $F_2(q, P, t) = f(q)P$ so that the generating function conditions become:

$$p = \frac{\partial F_2}{\partial q} = \frac{df}{dq}P \tag{8.34}$$

$$Q = \frac{\partial F_2}{\partial P} = f(q) \tag{8.35}$$

This allows us to represent the new position Q as an arbitrary function of the old position q. Observe that if we pick $f(q) = q$, then we obtain the identity. This means that the generating function $F_2 = qP$ achieves the identity transformation.

Infinitesimal transformation Consider the canonical transformation:

$$F_2 = qP + \epsilon G(q, P) \tag{8.36}$$

Note that as $\epsilon \to 0$, the transformation becomes the identity transformation. Then the generating function equations give:

$$p = \frac{\partial F_2}{\partial P} = P + \epsilon \frac{\partial G(q, P)}{\partial q} \tag{8.37}$$

$$Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{\partial G(q, P)}{\partial P} \tag{8.38}$$

Note that since p is close to P by the first equation, we can approximate $G(q, P) = G(q, p) + O(\epsilon)$, so that we obtain the approximate relations (note that P,Q are on the left and p,q are on the right):

$$P = p - \epsilon \frac{\partial G(q, p)}{\partial q} \tag{8.39}$$

$$Q = q + \epsilon \frac{\partial G(q, p)}{\partial p} \tag{8.40}$$

to first order.

But this looks like Hamilton's equations! G then tells how we should update p, q to obtain P, Q. Hence $H(q, p)$ is related to $G(q, p)$, and the infinitesimal canonical transformation $F_2 = qP + \epsilon H$.

Chapter 9

Week 9

9.1 Tuesday, 24 Nov 2015

9.1.1 Parametric Resonance

Define the vector

$$\vec{x} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ p_1 \\ p_2 \\ \vdots \end{pmatrix} \tag{9.1}$$

Then Hamilton's equations result in a set of first order differential equations that can be written:

$$\dot{\vec{x}} = \boldsymbol{A}\vec{x} \tag{9.2}$$

where $\boldsymbol{A}$ is a $2n \times 2n$ matrix that does not depend on time.

To include parametric effects, we need to let $\boldsymbol{A}$ depend on time:

$$\dot{\vec{x}} = \boldsymbol{A}(t)\vec{x} \tag{9.3}$$

and let $\boldsymbol{A}$ be periodic with period T.

Define $\vec{e}_j$ to be the solution to the matrix differential equation that has initial conditions that is $\vec{x}_0 = (0, \ldots, 1, 0, \ldots)$ where the 1 occurs in the ith position. Then the matrix where $\vec{e}_i$ are in the columns is the fundamental solution:

$$\vec{x}(t) = \boldsymbol{M}(t)\vec{x}(0) \tag{9.4}$$

The fundamental solution satisfies the matrix differential equation as well:

$$\boldsymbol{M}'(t) = \boldsymbol{A}(t)\boldsymbol{M} \tag{9.5}$$

Note further that $\boldsymbol{M}(0) = \boldsymbol{I}$, the identity matrix.

Floquet Theorem Statement:

$$\boldsymbol{M}(t+T) = \boldsymbol{M}(t)\boldsymbol{M}(T) \tag{9.6}$$

For multiple periods,

$$\boldsymbol{M}(t + jT) = \boldsymbol{M}(t)\boldsymbol{M}^j(T) \tag{9.7}$$

To examine stability, we hence want to examine the powers of $\boldsymbol{M}(T)$. The best way to raise matrices to power is to move into the diagonal basis. Note that:

$$\boldsymbol{M}^j(T) = (\boldsymbol{U}^{-1}\boldsymbol{D}\boldsymbol{U})^j = \boldsymbol{U}^{-1}\boldsymbol{D}^j\boldsymbol{U}$$

We want to examine the eigenvalues of $\boldsymbol{M}(T)$, which are the diagonal elements of $\boldsymbol{D}$.

Chapter 10

Week 10

10.1 Tuesday 1 Dec 2015

10.1.1 Equivalent statements for determining if a transformation is canonical

1. Phase space area is preserved Note that we require that the Jacobian determinant is unity for the volume in phase space to be the same:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$$

But the LHS is the Poisson bracket. Hence we require that:

$$[Q, P]_{q,p} = 1$$

Once the Poisson bracket of the coordinates is unity, we immediately have that the Poisson brackets in each of the coordinate systems is invariant:

$$[A, B]_{p,q} = [A, B]_{P,Q}$$

2. Canonical transformations are generated with a generating function
First consider a generating function of the first type. Then $F_1(q, Q, t)$ satisfies:

$$p = \frac{\partial F_1(q, Q, t)}{\partial q}, \quad \text{invert to get } Q(q, p, t)$$
$$P = -\frac{\partial F_1(q, Q, t)}{\partial Q}, \quad \text{substitute previous inversion to get } P(q, p, t)$$

Then the new Hamiltonian is:

$$K = H + \frac{\partial F_1}{\partial t}$$

Generating function for infinitesimal canonical transformation Consider $F_2(q, P) = qP + \epsilon G(q, P)$. Recall that $F_2(q, P) = qP$ is the identity transformation. The second term corresponds to a small perturbation. The generating function produces the transformation:

$$P = p - \epsilon \frac{\partial G(q, P)}{\partial q}$$
$$Q = q + \epsilon \frac{\partial G(q, P)}{\partial P}$$

We can replace P with p and group the quadratic and higher order corrections:

$$P = p - \epsilon\frac{\partial G(q,p)}{\partial q} + O(\epsilon^2)$$
$$Q = q + \epsilon\frac{\partial G(q,p)}{\partial p} + O(\epsilon^2)$$

Hamilton-Jacobi theory Suppose we can find a generating function $F_2(q, P, t)$ so that the new Hamiltonian is zero. Then in the new coordinate system, P, Q are constants. The time-dependent evolution is hence obtained from the map from $P, Q \to p, q$ from the generating function F_2.

Now we want:

$$K(q, P, t) = H(q, p, t) + \frac{\partial F_2(q, P, t)}{\partial t} = 0 \tag{10.1}$$

We want to write the equation using exactly two variables in $\{q, p, Q, P\}$. We choose to write the equation in terms of q, P.

Recall that the transformation equations for F_2 are:

$$p = \frac{\partial F_2(q, P, t)}{\partial q} \tag{10.2}$$
$$Q = \frac{\partial F_2(q, P, t)}{\partial P} \tag{10.3}$$

We can use the first equation to replace all instances of p. Then the new Hamiltonian is:

$$K(q, P, t) = H\left(q, \frac{\partial F_2(q, P, t)}{\partial q}, t\right) + \frac{\partial F_2(q, P, t)}{\partial t} \tag{10.4}$$

We now write:

$$S(q, t) = F_2(q, P, t) \tag{10.5}$$

for any fixed P (it is going to be a constant anyway). Then the equation we want to solve is:

$$H\left(q, \frac{\partial S(q, t)}{\partial q}, t\right) + \frac{\partial S(q, t)}{\partial t} = 0 \tag{10.6}$$

The previous equation is known as the Hamilton-Jacobi equation. If we can solve this equation for $S(q, t)$ we will obtain a whole family of possible functions corresponding to different values of P due to the constants of integration. To indicate that P is a constant, we write it as $P = \alpha$. Then:

$$p = \frac{\partial S(q, \alpha, t)}{\partial q} \tag{10.7}$$

Hamilton-Jacobi Equation in multiple dimensions

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{10.8}$$

which has $N + 1$ constants of integration. One of these constants will correspond to an additive constant to S. For the other N constants of integration, we denote them by $\alpha_k, k = 1, 2, \ldots, N$ and express the solution in terms of them:

$$S(q_k, \alpha_k, t) \tag{10.9}$$

Time-independent Hamiltonian If H is not dependent on time, its partial derivative with respect to time is zero. Then we can introduce:

$$S = -Et + W(q_k, \alpha_1, \ldots, \alpha_{N-1}) \tag{10.10}$$

which gives us the simplified Hamilton-Jacobi equation with $N-1$ constants of integration:

$$H\left(q_1, \ldots, q_N, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_{N-1}}\right) - E = 0 \tag{10.11}$$

The canonical transformation equations generated by the solution to the H-J theory is:

$$Q_l = \frac{\partial S(q_k, P_k, t)}{\partial P_l} \tag{10.12}$$

$$p_l = \frac{\partial S(q_k, P_k, t)}{\partial q_l} \tag{10.13}$$

Example of HJ Theory Consider 1D motion:

$$H = \frac{p^2}{2} + V(q) \tag{10.14}$$

First write down the H-J equation for time-independent Hamiltonians:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) - E = 0 \tag{10.15}$$

Rearranging,

$$W(q) = \int_{q_0}^{q} \sqrt{2E - 2V(x)}dx \tag{10.16}$$

where q_0 is a constant of integration and isn't important. Hence we have the formal solution:

$$S(q, P, t) = \int_{q_0}^{q} \sqrt{2(P - V(x))}dx - Pt \tag{10.17}$$

where we noted that the only relevant constant of integration is E, which we can write as $P = E$. Imposing the transformation equations,

$$Q = \frac{\partial S(q, P, t)}{\partial P} = \int_{q_0}^{q} \frac{1}{\sqrt{2(P - V(x))}}dx - t \tag{10.18}$$

which indeed is the formal solution to 1D motion because the integrand is the inverse velocity and $\frac{dx}{v} = dt$. The other coordinate is:

$$p = \frac{\partial S(q, P, t)}{\partial q} = \sqrt{2(P - V(x))} \tag{10.19}$$

which indicates that Q represents time while p represents the usual momentum.

Multi-dimensional system using Hamilton-Jacobi theory Consider the Hamiltonian in spherical coordinates with a special form for the potential:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U_r(r) + \frac{U_\theta(\theta)}{r^2} + \frac{U_\phi(\phi)}{r^2 \sin^2\theta} \tag{10.20}$$

The solutions are separable. That is, we can write:

$$S = W - Et = W_r(r) + W_\theta(\theta) + W_\phi(\phi) - Et \tag{10.21}$$

Substituting this ansatz into the H-J equation:

$$\frac{1}{2}\left(\frac{dW_r}{dr}\right)^2 + U_r(r) + \frac{1}{r^2}\left[\frac{1}{2}\left(\frac{dW_\theta}{d\theta}\right)^2 + U_\theta(\theta)\right] + \frac{1}{r^2 \sin^2\theta}\left[\frac{1}{2}\left(\frac{\partial W_\phi}{d\phi}\right)^2 + U_\phi(\phi)\right] = E \tag{10.22}$$

Now E is already a constant of integration. We need two other constants of integration. We note that the ϕ terms have to be a constant:

$$\frac{1}{2}\left(\frac{\partial W_\phi}{d\phi}\right)^2 + U_\phi(\phi) = \alpha_\phi \tag{10.23}$$

We pick the other integration constant to be:

$$\frac{1}{2}\left(\frac{dW_\theta}{d\theta}\right) + U_\theta(\theta) + \frac{\alpha_\phi}{\sin^2\theta} = \alpha_\theta \tag{10.24}$$

Then we want to solve the equation:

$$\frac{1}{2}\left(\frac{dW_r}{dr}\right)^2 + U_r(r) + \frac{\alpha_\theta}{r^2} = E \tag{10.25}$$

The solution to these differential equations fully specifies S.

10.1.2 Connection to QM

In operator terms, the QM Hamiltonian is:

$$\hat{H}\frac{\hat{P}^2}{2} + V(q) \tag{10.26}$$

and the momentum operator is:

$$\hat{P} = -i\hbar\partial_q \tag{10.27}$$

The wavefunction can be written as an exponential:

$$\psi \approx Ae^{iS(q,t)/\hbar} \tag{10.28}$$

we demand that the phase $S(q,t)$ changes rapidly with respect to time so that it has a high frequency, but that the frequency changes very slowly compared to the timescale of the frequency itself, so that we can write:

$$\partial_t\psi \approx A\frac{i}{\hbar}\frac{\partial S}{\partial t}e^{iS/\hbar} \tag{10.29}$$

The spatial derivative is also approximately:

$$\partial_q\psi \approx A\frac{i}{\hbar}\frac{\partial S}{\partial q}e^{iS/\hbar} \tag{10.30}$$

The second derivative of space is approximately (throwing out the derivative of $\frac{\partial S}{\partial q}$ since it is assumed to change very slowly):

$$\partial_q^2 \psi \approx A \left(\frac{i}{\hbar} \frac{\partial S}{\partial q} \right)^2 e^{iS/\hbar} \tag{10.31}$$

This is known as the WKB approximation. The Schrodinger equation is hence:

$$i\hbar \partial_t \psi = -A \frac{\partial S}{\partial t} e^{iS/\hbar} \tag{10.32}$$

$$-\frac{\hbar^2}{2} \partial_q^2 \psi \approx \frac{A}{2} \left(\frac{\partial S}{\partial q} \right)^2 e^{iS/\hbar} \tag{10.33}$$

$$\implies \frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 + V(q) + \partial_t S = 0 \tag{10.34}$$

but this is the Hamilton-Jacobi equation! Hence in the limit where the quantum mechanical de-Broglie wavelength goes to zero, the principal function S gives the phase of the wavefunction. This gives us a method of approximately solving the wave equation using the Hamilton-Jacobi principal solution.

To construct the full wavefunction, we need to superimpose plane wave solutions:

$$\psi = \int dE f(E) e^{iS(q,E,t)/\hbar} \tag{10.35}$$

where $f(E)$ is a weighting function that is peaked at E_0 with a phase that may vary linearly:

$$f(E) = g(E - E_0) e^{-i(E-E_0)\beta/\hbar} \tag{10.36}$$

where g is a function that is peaked at zero. We use the **method of steepest descent/method of stationary phase**. We note that the phase of the integrand is given by:

$$\frac{i}{\hbar} \left[S(q, E, t) - E\beta \right] \tag{10.37}$$

Hence for the values of q for which the wavefunction has non-zero value, we require that the phase is first order stationary with respect to E so that they add constructively when integrated across E:

$$\frac{\partial S(q, E, t)}{\partial E} = \beta \tag{10.38}$$

But this is the same relation as the canonical transformation:

$$Q = \frac{\partial S}{\partial P} \tag{10.39}$$

because $P = E$ is a constant of integration in the principal solution.

Relation to the Path Integral formulation of QM Recall that the propagator can be written as:

$$U(t) \propto \sum_{\text{all paths}} e^{iS/\hbar} \tag{10.40}$$

This suggests that the classical action is related to the principal function. Recall that the classical action is given by:

$$S_{cl} = \int_{t_1}^{t_2} L dt \tag{10.41}$$

Consider the time derivative of the principal function:

$$\frac{dS}{dt} = \frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial P}\dot{P} + \frac{\partial S}{\partial t} \tag{10.42}$$

Note that $\dot{P} = 0$ because the new coordinates are constants. Also note that since:

$$H + \frac{\partial S}{\partial t} = 0 \tag{10.43}$$

$$\frac{\partial S}{\partial q} = p \tag{10.44}$$

the time derivative is given by:

$$\frac{dS}{dt} = p\dot{q} - H = L \tag{10.45}$$

Hence the principal function is actually the classical action! Hence the generating function that converts the new Hamiltonian to zero is actually the classical action.

10.1.3 Action-Angle Variables

Consider motion that is periodic. We want to find the new variables I, ψ (action and action-angle respectively) so that the new Hamiltonian is a function of I alone and is not a function of ψ. Hence I and ψ are constants along the motion. The geometric meaning of I is the area enclosed by a closed trajectory in pq phase space (divided by 2π) and ψ varies from 0 to 2π as the system moves along a closed trajectory.

10.2 Thursday, 3 Dec 2015

10.2.1 Parametric resonance

Recall the Mathieu equation:

$$\ddot{\theta} + (a - 2q\cos 2t)\theta = 0 \tag{10.46}$$

The time dependent coefficient is essentially an external force feeding energy into the system. If the frequency of the external force matches the natural frequency of the system, the external force will feed energy into the system continually so that the system will become unstable.

10.2.2 More on Action-Angle Variables

Consider a time-independent Hamiltonian. Consider periodic motion. We want to find a canonical transformation $(q, p) \to (\psi, I)$ so that I is constant along the classical trajectory. Then we require that the Hamiltonian satisfy:

$$\dot{I} = -\frac{\partial H}{\partial \psi} = 0 \tag{10.47}$$

so that the Hamiltonian is just a function of I alone. Also, we require that along the trajectory:

$$\oint pdq = \oint Id\psi = I\oint d\psi \tag{10.48}$$

since the canonical transformations preserve the volume of phase space. For a closed periodic path, we can define I to be the area enclosed by the trajectory:

$$\frac{1}{2\pi}\oint pdq = I(q, p) \tag{10.49}$$

Hence ψ ranges from 0 to 2π over an orbit:

$$\oint d\psi = 2\pi \tag{10.50}$$

We define that the system when ψ increases by 2π should be the same system. Then we can wrap the plot of I against ψ onto a cylinder to match the 0 and 2π points.

We want to find a generating function $F_2(q, I)$ that performs this transformation.

Now by the transformation equations:

$$p(q, I) = \frac{\partial F_2(q, I)}{\partial q} \tag{10.51}$$

One such generating function can be obtained by directly integrating that transformation equation:

$$F_2(q, I) = \int_{q_0}^{q} p(x, I) dx \tag{10.52}$$

The other transformation equation is:

$$\psi = \frac{\partial F_2}{\partial I} = \int_{q_0}^{q} \frac{\partial p(x, I)}{\partial I} dx \tag{10.53}$$

We now want to verify that $\oint d\psi = \int \dot{\psi} dt = 2\pi$. The time derivative is:

$$\dot{\psi} = \dot{q} \frac{\partial p(q, I)}{\partial I} \tag{10.54}$$

$$\implies \int \dot{\psi} dt = \int \frac{\partial p(q, I)}{\partial I} dq = \frac{\partial}{\partial I} \int p dq = \frac{d}{dI}(2\pi I) = 2\pi \tag{10.55}$$

Observe further that:

$$\dot{\psi} = \frac{\partial H(I)}{\partial I} \tag{10.56}$$

which is the constant oscillation frequency:

$$\omega = \frac{\partial H(I)}{\partial I} = \dot{\psi} \tag{10.57}$$

Adiabatic invariance Consider a Hamiltonian parametrized by a single variable α. Then we write:

$$H(q, p, \alpha) \tag{10.58}$$

For a fixed α, $I(\alpha)$ will be constant and $\psi = \omega(\alpha)t$. Then the canonical transformation gives us:

$$q(\psi, I, \alpha), p(\psi, I, \alpha) \tag{10.59}$$

Let the canonical transformation come from the generating function $F_2(q, I, \alpha)$.

Now suppose α has time-dependence. We can still use the same generating function because the transformation does not depend on derivatives of α. The new Hamiltonian is now equal to:

$$\tilde{H} = H(q(\psi, I, \alpha), p(\psi, I, \alpha), \alpha) + \frac{\partial F_2}{\partial \alpha} \dot{\alpha} \tag{10.60}$$

because F_2 now has explicit time-dependence through α. Now note that the old Hamiltonian does not depend on ψ because $\dot{I} = -\frac{\partial H}{\partial \psi} = 0$. Hence we can write the new Hamiltonian as:

$$\tilde{H} = H(I, \alpha) + \frac{\partial F_2}{\partial \alpha}\dot{\alpha} \tag{10.61}$$

By Hamilton's equations,

$$\dot{I} = -\frac{\partial}{\partial \psi}\left(\frac{\partial F_2}{\partial \alpha}\dot{\alpha}\right) \tag{10.62}$$

$$\dot{\psi} = \omega(I, \alpha) + \frac{\partial}{\partial I}\left(\frac{\partial F_2}{\partial \alpha}\dot{\alpha}\right) \tag{10.63}$$

Now the functional dependence of the generating function is $F_2(q(I, \psi, \alpha), I, \alpha)$. F_2 is hence a periodic function in ψ. This means that $\dot{I}$ is a derivative of a periodic function in ψ. Hence when integrated across a period, I will be constant. Then we can write:

$$\dot{I} = \frac{\partial}{\partial \psi} f(\psi, I, \alpha)\dot{\alpha} \tag{10.64}$$

$$\dot{\psi} = \omega(I, \alpha) + g(\psi, I, \alpha)\dot{\alpha} \tag{10.65}$$

Now consider time evolution when ψ increases by 2π. Let ΔI be the change in I over that increase. We want to show that $\frac{\Delta I}{2\pi}$ is of small order, that is, it is of order $\dot{\alpha}^2$ and $\ddot{\alpha}$. Then:

$$\frac{dI}{d\psi} = \frac{\dot{I}}{\dot{\psi}} = \frac{1}{\omega_0}\frac{\partial f}{\partial \psi}\dot{\alpha} + O(\dot{\alpha}^2) \tag{10.66}$$

Now we can also approximate:

$$f(\psi, I, \alpha) \approx f(\psi, I_0, \alpha_0) + O(\dot{\alpha}) \tag{10.67}$$

and hence we can throw the terms into the higher order collection terms:

$$\frac{dI}{d\psi} = \frac{\partial f(\psi, I_0, \alpha_0)}{\partial \psi}\frac{1}{\omega_0}\dot{\alpha}_0 + O(\dot{\alpha}^2, \ddot{\alpha}) \tag{10.68}$$

Integrating both sides from $\psi = 0$ to $\psi = 2\pi$, we hence have:

$$\int_0^{2\pi} \frac{dI}{d\psi} d\psi = O(\dot{\alpha}^2, \ddot{\alpha}) \tag{10.69}$$

because $f(\psi, I_0, \alpha_0)$ is purely periodic in ψ. This is the concept of adiabatic invariance.

10.2.3 Review of term