Ph12a Class Notes

Lim Soon Wei Daniel

Chapter 1

Week 1

1.1 Tuesday 30 Sept 2014

General potential: Write the Taylor expansion of a general potential: $U(x) = U(x_0) + \frac{dU}{dx}\Big|_{x_0}(x-x_0) + \frac{1}{2}\frac{d^2U}{dx^2}\Big|_{x_0}(x-x_0)^2 + \ldots$ When the net force is zero, the first derivative is zero. Note hence that if $\frac{d^2U}{dx^2} > 0$ then the local equilibrium is stable and $k = \frac{d^2U}{dx^2}\Big|_{x_0}$.

General solution to oscillatory motion: $x(t) = \frac{v(t_0)}{\omega}\sin(\omega t) + x(t_0)\cos(\omega t)$. In exponential form, $x(t) = Ce^{\beta t}, \beta = \pm i\omega$. Alternatively, $z(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}, x(t) = Re\,(z(t))$. Note that the imaginary part can be used too.

Elastic rod pendulum: Consider the Young's modulus: $Y = \frac{F/A}{\Delta x/x_0}$ (stress over strain). Typical Y values: $6\times 10^{10} Pa$ for aluminium. Yield strength is much lower: $2\times 10^8 Pa$. Note that the yield strength sets the limit of the maximum value of the displacement Δx.

1.2 Thursday 2 Oct 2014

Excitation of normal modes Note that if you only excite one degree of freedom in a system with multiple normal modes, you actually excite more than one normal mode (perhaps all?).

General linear system $a\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + cx(t) = F(t)$. Write as a linear differential operator $\mathcal{L}(x)$. Note that if there are two orthogonal solutions to the homogenous differential equation (i.e. $f(t) = 0$), then any linear combination of the two solutions is also a solution. Note also that by linearity, $\mathcal{L}(ax_1 + bx_2) = a\mathcal{L}(x_1) + b\mathcal{L}(x_2)$. If $\mathcal{L}(x) = f_1(t)$ has a solution $x_1(t)$ for the non-homogenous differential equation and $\mathcal{L}(x_2) = f_2(t)$ has another solution $x_2(t)$, and we have the non-homogenous differential equation $\mathcal{L}(x) = Af_1(t) + Bf_2(t)$, then $Ax_1(t) + Bx_2(t) + ax_0(t)$ is a solution (a is arbitrary). Note that $x_0(t)$ is the solution to the homogenous differential equation $\mathcal{L}(x_0) = 0$. Now if we have the solution for $f(t) = \sin(\omega t)$, then we can solve the differential equation for any $f(t)$ that can be decomposed through Fourier analysis.

Two independent harmonic oscillators Consider $x_1(t) = A_1\cos(\omega_1 t + \phi_1) = Re\left(A_1 e^{i(\omega_1 t + \phi_1)}\right) = Re(z_1(t))$ and $x_2(t) = A_2\cos(\omega_2 t + \phi_2)$ similarly.

Case 1 $A_1 = A_2 = A$, $\omega_1 = \omega_2 = \omega$. Define $\bar{\phi} = \frac{\phi_1 + \phi_2}{2}$. Then $z_1 + z_2 = Ae^{i(\omega t + \bar{\phi})}\left[e^{-i(\phi_2 - \phi_1)/2} + e^{i(\phi_2 - \phi_1)/2}\right] = 2A\cos\left(\frac{\phi_2 - \phi_1}{2}\right)e^{i(\omega t + \bar{\phi})}$.

Recall that $\cosh(i\theta) = \cos\theta$ **and** $\sinh(i\theta) = i\sin\theta$

Case 2 Consider $A_1 = A_2 = A$ and $\phi_1 = \phi_2 = 0$ and $\omega_1 \neq \omega_2$. Enter the co-rotating frame with z_1. Then the angle between z_1 and z_2 is $(\omega_2 - \omega_1)t$. Define $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ and $\delta\omega = \omega_2 - \omega_1$. Then write $z_1 + z_2 = Ae^{i\omega t}\left[e^{-i(\omega_2 - \omega_1)t/2} + e^{i(\omega_2 - \omega_1)t/2}\right] = 2A\cos\left(\frac{(\omega_2 - \omega_1)t}{2}\right)e^{i\omega t} = 2A\cos(\delta\omega t/2)e^{i\omega t}$. If $\omega_1 \approx \omega_2$, but not equal, then $\delta\omega << \bar{\omega}$. Note that the amplitude modulating factor $\cos(\delta\omega t/2)$ has a period of $\frac{2\pi}{\delta\omega}$ because the beat has a maximum even when the value of the cosine is at the negative peak. Hence the effective period is half that of $\frac{2\pi}{\delta\omega/2}$, which is $\frac{2\pi}{\delta\omega}$.

Coupled oscillators Consider two masses m_1, m_2 and three springs (all k) on a frictionless surface. Let the positions x_1 and x_2 be measured with respect to the individual positions of equilibrium. Then the equations of motion are

$m_1\ddot{x_1} = -kx_1 + k(x_2 - x_1)$ and $m_2\ddot{x_2} = -kx_2 + k(x_1 - x_2)$. Write in general, $m_j\ddot{x_j} = f_j = \sum_{l=1}^{n} k_{jl}x_l$ where $j = 1, \ldots, N$ are the masses and the l are couplings. Now the individual forces f_j are the derivatives of potentials $f_j = \frac{-\partial V}{\partial x_j}$ if they are conservative. Hence the potential is a function of all the masses $V(x_1, x_2, \ldots, x_N)$, and is quadratic in all the x. Then $\frac{\partial f_j}{\partial x_l} = -\frac{\partial^2 V}{\partial x_j \partial x_l} = -\frac{\partial^2 V}{\partial x_l \partial x_j}$. Hence $k_{jl} = k_{lj}$. Call k_{jl} the K matrix. We look out for normal modes, which is the coupled motion of the components of the system that are moving with the same angular velocity and phase. We want oscillation equations, hence look out for solutions of the form $x_1 = A_1 \cos(\omega t + \phi)$ and $x_2 = A_2 \cos(\omega t + \phi)$. Hence $\ddot{x_1} = -A_1\omega^2 \cos(\omega t + \phi) = -\omega^2 x_1$ and similarly for $\ddot{x_2}$. We substitute these (replace the double dots) into the equations of motion to obtain $-\omega^2 x_1 + (k/m_1)x_1 - (k/m_1)(x_2 - x_1) = 0$ and $-\omega^2 x_2 + (k/m_2)x_2 + (k/m_2)(x_2 - x_1) = 0$. In matrix form, $\begin{pmatrix} -\omega^2 + 2k/m_1 & -k/m_1 \\ -k/m_2 & -\omega^2 + 2k/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Taking $m_1 = m_2 = m$ and setting the determinant to zero, solving for $\omega^2 = k/m$ and $3k/m$. Sub these angular frequency values back into the matrix to obtain values for A_1 and A_2.

Generalized to N oscillators The K matrix is symmetric and has dimensions $N \times N$. Define the mass matrix $M = diag(m_1, m_2, \ldots, m_N)$ and the amplitude column vector $A = (A_1, A_2, \ldots, A_N)$. Then we have that $M\omega^2 A = -KA$ or $M\ddot{x} = -Kx$ or $(M^{-1}K - \omega^2 I)A = 0$. Hence we seek the eigenvalues of the $M^{-1}K$ matrix to find ω. For systems without damping, the roots of the characteristic polynomial will be positive and real. The free motion of these N degrees of freedom can always be written as a linear superposition of these N normal mode solutions.

Chapter 2

Week 2

2.1 Tuesday 7 Oct 2014

Infinite number of oscillators We let the distance between masses go to zero. We will obtain a solid elastic body with a mass per unit length of μ.

A two-mass example Consider two beads of mass $m/2$ each on a massless string with total length L. Then the mass per unit length $\mu = \frac{m}{L}$. Let the masses oscillate in the transverse direction by an amount y_1 and y_2 for masses m_1 and m_2 respectively. Let the string be under tension T. We expect two normal modes. For small oscillations, the restoring force in the transverse direction is proportional to the displacement y. Writing the equations of motion for the fundamental mode:

$$\frac{m}{2}\ddot{x} = -T\cos\theta + T \approx 0$$
$$\frac{m}{2}\ddot{y} = -T\sin\theta + 0 \approx -T\frac{3y}{L}$$

Hence we have that $\omega_1 = \sqrt{\frac{6T}{mL}}$.

Now write the equations of motion for the other normal mode (with the angle of the middle string being $\theta', \tan\theta' = \frac{6y}{L}$).

$$\frac{m}{2}\ddot{x} = -T\cos\theta + T\cos\theta' \approx 0$$
$$\frac{m}{2}\ddot{y} = -T\sin\theta - T\sin\theta' \approx -T\frac{-9y}{L}$$

Hence we have that $\omega_2 = \sqrt{\frac{18T}{mL}}$.

Continuous system Consider a system with total mass M on a string of length L. Then we can think of the system as having N masses of mass $\frac{M}{N}$ each separated by $\frac{L}{N+1}$. Then a chuck of the string has mass $\mu\delta x$ and is under the influence of tension T on both sides, acting with angle θ on the left and θ' on the right. Write $\theta' = \theta - \delta\theta$. Writing the equations of motion for that bit of string,

$$\mu\delta x\ddot{x} = -T\cos\theta + T\cos(\theta') \approx 0$$
$$\mu\delta x\ddot{y} = -T\sin\theta + T\sin(\theta') = -T(\sin\theta - \sin(\theta - \delta\theta)) \approx -T\delta\theta$$

Rewriting,

$$\ddot{y} = -\frac{T}{\mu}\frac{\delta\theta}{\delta x}$$

But $\delta\theta = \left.\frac{\partial y}{\partial x}\right|_{x_0} - \left.\frac{\partial y}{\partial x}\right|_{x_0+\delta x}$, so $\ddot{y} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$. We note that $\frac{T}{\mu}$ has units of velocity squared. The speed of sound or travelling waves in a massive string is $\sqrt{\frac{T}{\mu}}$.

Normal modes in the continuous system The normal modes have the form $f(x)e^{i(\omega t+\phi)}$. Substituting this into the wave equation, we have that $\frac{\partial^2 f}{\partial x^2} + \left(\frac{\mu}{T}\omega^2\right) f = 0$. The boundary conditions of this is that $f(0) = 0$ and $f(L) = 0$. We know that $f(x)$ will be of the form $f(x) = A\cos\left(\sqrt{\frac{\mu}{T}}\omega x + \alpha\right)$. Substituting in the boundary conditions, we obtain $\alpha = \pm\pi/2$, so we use the sine instead. We also obtain from $f(L) = 0$ that $\sqrt{\frac{\mu}{T}}\omega L = n\pi, n \in \mathbb{Z}$. Hence $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$. We see that the angular frequencies are quantised. We can also calculate the wavelength $\lambda_n = \frac{2L}{n}$. Call $k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$ the angular wavenumber. Hence we obtain the dispersion relation $\omega_n = \sqrt{\frac{T}{\mu}}k_n$.

2.2 Thursday, 9 Oct 2014

Normal Modes of continuous system Recall that the normal modes of a string clamped down on both ends look like $y(x,y) = A\cos(\omega t + \phi)\sin(\sqrt{\frac{\mu}{T}}\omega x)$. Also recall that there exists an upper limit to the normal mode frequencies in free motion. Lower limit to wavelength in discrete systems is 2x size of smallest lumped element.

Fourier's Theorem Any piecewise continuous function defined between $[0, L]$ in 1-D can be expressed as an infinite sum of sines and cosines with wavelengths of $\frac{2L}{n}$. Also works for an infinite domain with periodicity (wavelength $2L$). Just define the function to by reflected about the point $x = L$ over the domain $[L, 2L]$ and so on. Define $y(x, t = 0) = f(x)$ to be the position initial condition. Write $f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$. Note that $\int_0^{2L} dx \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$ and $\int_0^{2L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \int_0^{2L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = L\delta_{mn} = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$. Now we multiply $f(x)$ with the sines and cosines to obtain: $\int_0^{2L} f(x)\sin\frac{m\pi x}{L}dx = a_m L$ and $\int_0^{2L} f(x)\cos\frac{m\pi x}{L}dx = b_m L$. We also integrate $f(x)$ itself over the interval to obtain $\int_0^{2L} f(x)dx = \frac{b_0}{2}2L = b_0 L$. Hence we have the Fourier coefficients:

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$
$$a_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx$$
$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{m\pi x}{L}dx$$
$$b_0 = \frac{1}{L}\int_0^{2L} f(x)dx$$

Chapter 3

Week 3

3.1 Tuesday 14 Oct 2014

Longitudinal oscillation Consider a cantilever with one fixed end on the left. Consider a small slice of the rod between x and $x + \delta x$. Then the left side of the slice is pulled with force f_1 to the left and the right side is pulled with for f_2 to the right. Let the function $\psi(x,t)$ represent the deviation of the rod from equilibrium at x. Then the equations of motion are: $f_2 - f_1 = (M\frac{\delta x}{L})\ddot{\psi}$ where M is the mass of the entire rod with length L. Now recall the Young's modulus $Y = \frac{F/A}{-\delta\psi/\delta x}$. Then the stress is given by $-Y\frac{\partial\psi}{\partial x}$. Hence we have that $\frac{f_2-f_1}{A} = Y\frac{\partial\psi}{\partial x}(x+\delta x) - Y\frac{\partial\psi}{\partial x}(x)$, which we can write to form $\frac{f_2-f_1}{A} = Y\delta x\frac{\partial^2\psi}{\partial x^2}$. Comparing this to the equation of motion, we have that $\frac{M}{AL}\delta x\ddot{\psi} = Y\delta x\frac{\partial^2\psi}{\partial x^2}$. Note that the density $\rho = \frac{M}{AL}$, hence the velocity (comparing to the wave equation) is given by $v = \sqrt{\frac{Y}{\rho}}$.

3.2 Thursday 16 Oct 2014

Mechanical Band-pass filter Note that the lumped oscillators have a fixed minimum frequency and a fixed maximum frequency (the zig-zag mode). Hence it acts as a band-pass filter since it doesn't transmit energy outside this band.

Driven damped oscillator General equation of motion: $m\ddot{x} = -kx - m\Gamma\dot{x} + F(t)$. In real systems, there may be damping effects proportional to higher powers of $\dot{x}$. The constant proportional to the second derivative of x is called the effective mass. k is the effective spring constant, and Γ is the effective damping term. We want solutions of the form $x(t) = Ae^{i\omega t}$, and substituting this, we obtain $(-\omega^2 + i\omega\Gamma + k/m)Ae^{i\omega t} = F(t)/m$. Note that if the equation is non-linear then we cannot use the complex exponentials. Alternatively, we can perform a Laplace transform on the differential equation by writing $x = x_0e^{\alpha t}$, for complex α, hence we obtain $\dot{x} = \alpha x, \ddot{x} = \alpha^2 x$. So we have $\alpha^2 x + \alpha\Gamma x + \omega^2 x_0 e^{\alpha t} = 0$. For non-zero x_0, we have that $\alpha^2 + \alpha\Gamma + \omega^2 = 0$, which can be solved using the quadratic equation to obtain $\alpha = \frac{-\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$. Case 1 (underdamped), the thing inside the square root sign is negative, so $\omega^2 > \frac{\Gamma}{4}$, then $\alpha = \frac{-\Gamma}{2} \pm i\omega_1, \omega_1 = \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$. Then $x(t) = x_0e^{\alpha t} = x_0e^{-\Gamma/2}\left(ae^{i\omega_1 t} + be^{-i\omega_1 t}\right)$ where a and b are determined by the initial conditions. Case 2 (over damped), we have that $\frac{\Gamma^2}{4} > \omega_0^2$ then α is purely real. Call $\beta = \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}, \alpha = \frac{-\Gamma}{2} \pm \beta$ so the solutions are $x(t) = Ae^{-(\Gamma/2+\beta)t} + Be^{-(\Gamma/2-\beta)t}$. Note that the over damped motion may cross the zero axis at most once (overshoot). Case 3 (critical damping) when $\frac{\Gamma^2}{4} = \omega_0^2$. Then we have that $\alpha = \frac{-\Gamma}{2}$ and $x(t) = Ae^{-\Gamma t/2} + Bte^{-\Gamma t/2}$. Critical damping goes to zero faster than overdamping.

General driving force We write $f(t) = \sum_n [A\cos(\omega_n t) + B\sin(\omega_n t)]$ if $f(t)$ is periodic with period $T = \frac{2\pi}{\omega_n}$. However, if the period is infinite, then we replace the sum with an integral $\sum_n \to \int d\omega$.

Single sinusoidal driving force Let $f(t) = f_0\cos(\omega t)$. We note that solutions will be oscillating only at ω in equilibrium. Then we write $z = z_0e^{i\omega t}$ and obtain $(-\omega^2 + i\omega\Gamma + \omega_0^2)z_0e^{i\omega t} = \frac{f_0}{m}e^{i\omega t}$. Hence we have that $z_0 = \frac{f_0/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}$.

Chapter 4

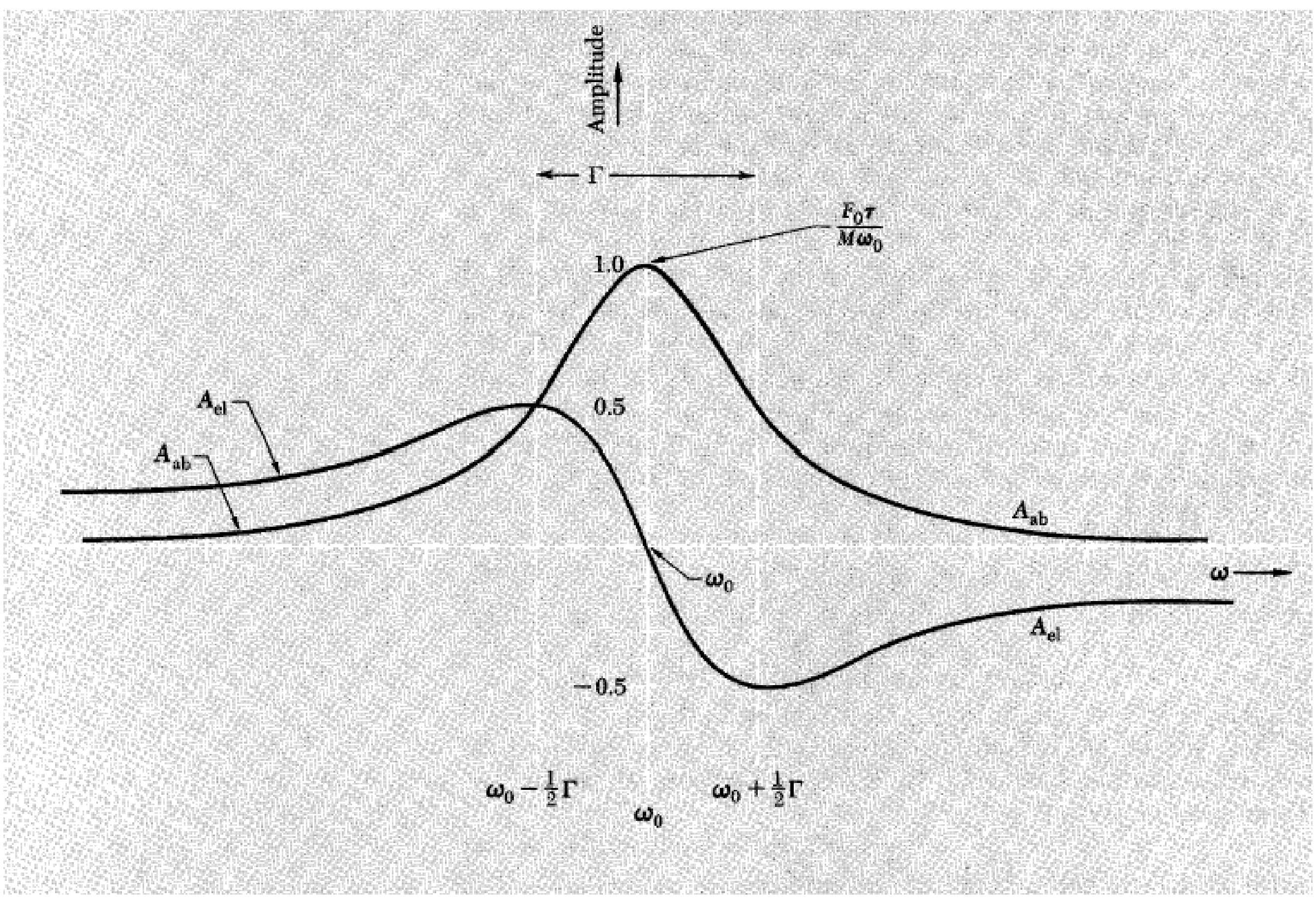
Week 4

4.1 Tuesday 21 Oct 2014

Single sinusoidal driving force We resume from the previous week where we obtained that the solution $z = z_0 e^{i\omega t}$ to the damped driven harmonic oscillator has $z_0 \frac{f_0/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}$, which can be written as $\frac{f_0}{m}\left(\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}\right) - i\frac{f_0}{m}\frac{\omega\Gamma}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}$. Call the real part the elastic amplitude and the imaginary part the absorptive amplitude. Hence we have:

$$A_{elastic} = \frac{f_0}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}$$
$$A_{abs} = \frac{f_0}{m}\frac{\omega\Gamma}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}$$

Plotting these amplitudes on the same graph, we obtain:



We can also write $z_0 = |A|e^{-i\delta}$ where $\tan\delta = \frac{\omega\Gamma}{\omega_0^2-\omega^2}$.

At $\omega = \omega_0$ we have that the absorptive amplitude is large and the elastic amplitude goes to zero. Note that the absorptive amplitude DOES NOT peak at $\omega = \omega_0$. This peak is shifted based on the damping Γ.

We also note that the power delivered by the driving force to the system is $P(t) = \vec{F}(t) \cdot \vec{v}$. The average power over one cycle peaks at $\omega = \omega_0$ and is equal to $\frac{f_0^2}{2m\Gamma}$ there. We can write $< P > (\omega) = \frac{f_0^2}{2m\omega} \frac{\omega^2 \Gamma^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$. The half-peak power points occur at $\omega_{1/2} = \sqrt{\omega_0^2 + (\Gamma/2)^2} \pm (\Gamma/2)$. Hence the full width at half maximum (FWHM) is $\Delta\omega = \Gamma$. Note that if f is suddenly removed, the amplitude decays as $e^{-\Gamma t}$ so the characteristic lifetime is $\Delta t = \tau = \frac{1}{\Gamma} = \frac{1}{FWHM}$. We hence note that $\Delta\omega \Delta t = 1$.

Approximations If $\omega \approx \omega_0$ then we can write $< P >= P_0 \frac{1}{1 + (\frac{\omega_0 - \omega}{\Gamma/2})^2}$, which we can write as $P_0 \frac{d}{dx}(\tan^{-1} x)$ at $x = \frac{\omega_0 - \omega}{\Gamma/2}$.

Quality Factor The quality factor Q is defined to be ω_0/Γ, the ratio of the peak frequency to the FWHM. This is also equal to the number of oscillations at ω_0 over a time $\Delta t = \frac{1}{\Gamma}$.

4.2 Thursday 23 Oct 2014

General excitatory force Write $f(t) = \int d\omega \tilde{f}(\omega) \cos(\omega t)$, where $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$.

Travelling waves in 1D Recall the classical wave equation: $\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}$. In the normal mode for fixed-end boundaries, we write $y_n = A_n \sin(k_n x) \cos(\omega_n t + \phi), k_n = \frac{n\pi}{L}, \omega_n = \sqrt{\frac{T}{\mu}} k_n$. We can write this using $2 \sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$. Hence we have $y_n(x, y) = \frac{A_n}{2} [\sin(k_n x - \omega_n t) + \sin(k_n x + \omega_n t)]$. Notice that each of the terms is a travelling wave, one travelling in the x direction $(k_n x - \omega_n t)$ and one in the -x direction $(k_n x + \omega_n t)$. Write $k_n x \pm \omega t = k_n (x \pm v_p t)$, $v_p = \omega_n / k_n$, the phase velocity.

Travelling waves in 3D Write $\psi(\vec{r}, t) = A \sin(\vec{k} \cdot \vec{r} - \omega t) = \Im[A e^{i(\vec{k} \cdot \vec{r} - \omega t)}]$. Write $\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$. Note that the wave will propagate in the $+\hat{k}$ direction with velocity given by $\frac{\omega}{|\vec{k}|} \hat{k}$.

Circular Waves Note that since the energy in a wave is proportional to the amplitude square, for the energy of a spreading wave to be conserved, the amplitude of the circular wave will have to decrease as it spreads outward. We hence have $|A|^2 2\pi r dr$ =constant so the amplitude falls off as $\frac{1}{\sqrt{r}}$. In 3D, we have $|A|^2 4\pi r^2 dr$ =constant so the amplitude falls off as $\frac{1}{r}$.

Dispersion relations

- Continuous string: $\omega = vk, v = \sqrt{\frac{T}{\mu}}$
- Beaded string, transverse: Assume separation a, individual bead mass m, $\omega = \frac{4T}{ma} \sin \frac{ka}{2}$.
- Beaded string, transverse, long wavelength approximation: Let ka be small. Then $\omega = \frac{4T}{ma} \frac{ka}{2} = \sqrt{\frac{T}{\mu}} k$. Higher order Taylor approximations will have corrections proportional to higher powers of k.
- Real piano wire: $\omega^2 = \frac{T}{\mu} k^2 + \alpha k^4$. Note at short wavelengths, and if $\alpha > 0$, then the effective tension appears to be larger than usual (piano wire becomes stiffer when it is bent over short distances).
- Mass on spring, longitudinal oscillation: $\omega = \sqrt{\frac{4K}{m}} \sin \frac{ka}{2}$. In the long wavelength limit, we have $\omega = \sqrt{\frac{4K}{m}} \frac{ka}{2} = \sqrt{\frac{Ka}{m/a}} k$ where K is the spring constant and k is the wavenumber. Note that m/a plays the role of mass per unit length. Hence Ka plays the role of the tension.
- LC transmission line: $\omega = \sqrt{\frac{4}{LC}} \sin \frac{ka}{2}$. In the long wavelength limit, we have $\omega = \sqrt{\frac{a^2}{LC}} k = \frac{1}{\sqrt{(L/a)(C/a)}} k$ where we have the inductance per unit length L/a and capacitance per unit length C/a.
- Coupled pendulum: $\omega^2 = \frac{g}{l} + \frac{4K}{m} \sin^2 \frac{ka}{2}$.
- Ionized plasma: Consider an ionised, overall electrically neutral plasma in a capacitor experiencing an electric field $E(t)$. Then we have $\omega^2 = \omega_p^2 + c^2 k^2$, where $\omega_p = \sqrt{\frac{4\pi N_e e^2}{m_e}}$ is the plasma frequency, c is the speed of light.
- Continuous medium, longitudinal oscillations: $\omega = \sqrt{\frac{Y}{\rho}} k$. For air, $Y = K_{ad}$, where $K_{ad} = \gamma P$ is the adiabatic compressibility, γ is the adiabatic constant, and P is the pressure.

Electromagnetic Waves - Maxwell's Equations Recall that we have (in CGS units)

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = \frac{-1}{c}\dot{\vec{B}}$$
$$\nabla \times \vec{E} = \frac{1}{c}\dot{\vec{E}} + \frac{4\pi}{c}\vec{J}$$

In a vacuum, we have $\rho = 0$ and $\vec{J} = 0$, so:

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = \frac{-1}{c}\dot{\vec{B}}$$
$$\nabla \times \vec{E} = \frac{1}{c}\dot{\vec{E}}$$

Taking the curl of the curls, we have:

$$\nabla \times (\nabla \times \vec{E}) = \frac{-1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{B}) = \frac{-1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$
$$\nabla \times (\nabla \times \vec{B}) = \frac{-1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}$$

But the curl of the curl is given by $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2\vec{v}$. Hence we have a 3D wave equation for the electric and magnetic fields:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$
$$\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}$$

Hence we have the velocity of propagation is c. We can write $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, where we note that the amplitude is a 3-vector.

Chapter 5

Week 5

5.1 Tuesday 28 Oct 2014

Travelling wave solutions of E and B fields $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i\vec{k}\cdot\vec{r}-\omega t}$ with $|k| = \frac{2\pi}{\lambda}$, $\hat{k}$ is the direction of propagation of the wave.

Relation between E and B fields Recall that $\nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}$ but $\nabla \times \vec{E} = i\vec{k} \times \vec{E}_0 e^{i\vec{k}\cdot\vec{r}-\omega t}$ so $\frac{-1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c}\vec{B}_0 e^{i\vec{k}\cdot\vec{r}-\omega t}$. But since $|\vec{k}| = \omega/c$, we have that $|\vec{E}_0| = |\vec{B}_0|$ in a vacuum (CGS UNITS). Observe that $\hat{k}, \hat{E}, \hat{B}$ form a right-handed coordinate system because $\vec{B}$ is perpendicular to both $\vec{k}$ and $\vec{E}$: $\hat{E} \times \hat{B} = \hat{k}$. Energy propagation is given by the Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}$, which has units of energy per unit area per unit time (intensity).

Plane Waves Consider $\hat{k} = \hat{z}$. Then we have $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(kz-\omega t)}$ such that they fill all of space.

EM waves in Matter Matter contains charged particles. Electrons, having a much higher charge/mass ratio, respond a lot to passing EM waves. We ignore large nuclei and ions for now. We also ignore tightly-bound electrons. Hence we are only interested in free electrons and weakly-bound electrons. Possible scenarios:

- Electrons vibrate with velocities in phase with the electric field, the field does work and energy will be absorbed. Example: conductors.
- If the electrons vibrate with positions in phase with the electric field, the velocities are $\pi/2$ out of phase with the field and no work is done on the electron so the wave propagates. Example: dielectrics.

Dielectric constant Consider a neutral dielectric. Then we have $\nabla \cdot \vec{E} = 0, \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \frac{\epsilon}{c}\frac{\partial E}{\partial t}$ (note 4th equation different from vacuum). Let $\epsilon = 1$ for vacuum. Now when we write the wave equation, we have the dispersion relation:

$$v = \omega/k = c/\sqrt{\epsilon} = c/n$$

where $n = \sqrt{\epsilon}$ is the index of refraction. Hence the power $|\vec{S}| = \sqrt{\epsilon}\frac{c}{4\pi}|\vec{E}|^2$.

Boundary conditions Note that ω is continuous across boundaries. However, $k = \frac{2\pi}{\lambda}$ changes discontinuously across the boundary. Note that $\lambda = \frac{1}{n}\lambda_{vac}$ so $|\vec{k}| = n|\vec{k}_{vac}|$.

Polarization of atoms We note that as an electric field propagates through a medium, the loosely bound electrons are displaced by $\vec{d}$ and hence the atom is polarised with polarisation $\vec{P}$, dipole moment $q\vec{d}$ per unit volume. Then the total electric field is given by $\vec{E}_{tot} = \vec{E}_{extreme}(t) - 4\pi\vec{P}(t)$. We can write $\vec{P}(t)$ =number of electrons per unit volume times charge per electron times $\vec{X}(t)$, the displacement along the $\vec{E}(t)$ direction. We model the atomic response for $x(t)$ as a simple harmonic oscillator with restoring force and dissipation (damping):

$$m_e\ddot{x} = -m\omega_0^2 x - m_e\Gamma\dot{x} - eE(t) \tag{5.1}$$

since the charge of the electron is $-e$. We note that the external electric field has angular frequency ω. The solutions are:

$$x(t) = A_{el}\cos(\omega t) + A_{ab}\sin(\omega t) \tag{5.2}$$

we note that the position elastic part is in-phase with the electric field and the absorptive part is out of phase with the electric field (so that the velocity is in phase with the electric field, allowing absorption). Recall that $A_{el} = \frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}$, and $A_{ab} = \frac{F_0}{m}\frac{\omega\Gamma}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}$. Note that near the resonant frequency, the elastic part goes to zero, the absorptive part peaks, so the electric field is absorbed and not transmitted. This happens in conductors. On the other hand, if ω is far from the resonant frequencies, the converse happens and the wave propagates.

Far from resonance Take $A_{ab} \approx 0$ and $|\omega_0^2 - \omega^2| >> \omega_0\Gamma$, the width of the resonance, then $A_{el} \approx \frac{eE_0}{m_e}\frac{1}{\omega_0^2-\omega^2}$. Then the dielectric constant $\epsilon = \frac{E_{ext}}{E_{tot}} = 1 + \frac{4\pi P}{E_{tot}} \approx 1 + \frac{4\pi N\cdot e\cdot x(t)}{E_0} \approx 1 + \frac{4\pi Ne^2}{m_e}\frac{1}{\omega_0^2-\omega^2}$. For a large number of electrons, and for ω far away from all resonances, we write $\sum_{i=1}^{N_r}\frac{1}{\omega_{0,i}^2-\omega^2} \approx \frac{N_r}{\overline{\omega_0^2}-\omega^2}$. Hence we write $\epsilon = 1 + \frac{4\pi Ne^2}{m_e}\frac{N_r}{\overline{\omega_0^2}}(1 + \frac{\omega^2}{\overline{\omega_0^2}} + \ldots)$. Hence we see that the index of refraction $n = \sqrt{\epsilon}$ increases with frequency for $\omega << \sqrt{\overline{\omega_0^2}}$.

Plasma Oscillations Consider free electrons now (e.g. plasma or metal). Now there are no restoring forces, so the resonant frequencies are zero. Hence we let $\overline{\omega_0^2} = 0, N_r = 1$, and we have $\epsilon = 1 - \frac{4\pi Ne^2}{m_e}\frac{1}{\omega^2}$, note the minus sign. We write $\omega_p = \sqrt{\frac{4\pi Ne^2}{m_e}}$. We note that if $\omega > \omega_0^2$ then $\epsilon < 0$ so the index of refraction becomes imaginary. Then instead of writing $k = nk_{vac}$, we write $k = i|n||k_{vac}|$ so $e^{i(i|n||k_{vac}|z-\omega t)} = e^{-|n||k_{vac}|z}e^{-i\omega t}$, and we have an exponentially decreasing wave with position, an evanescent wave. Note that there will be no phase velocity, and hence no propagation.

Alternatively, consider a volume of free charged particles displaced by $x(t)$, find the induced electric field, and take it to be the restoring force.

Plasma Dispersion Relation Consider from the previous paragraph $\omega^2\epsilon = \omega^2 n^2 = \omega^2 - \omega_p^2$. The dispersion relation for such a medium is $\omega = ck/n$, so we substitute to obtain $\omega^2(ck/n)^2 = \omega^2 - \omega_p^2$ so we obtain $\omega^2 = c^2k^2 + \omega_p^2$, the dispersion relation for a plasma. Note that ω_p acts like a minimum frequency. When the driving ω is less than ω_p, then k becomes imaginary and we obtain an evanescent wave.

Dielectric Dispersion Relation Recall that ω_0^2 is the restoring force per unit distance per unit mass. $\omega_0^2 = \frac{e^2}{a}/\frac{a}{m_e} = \frac{e^2 m_e}{a^2}$.

5.2 29 Oct 2014 Recitation

Traveling waves Recall the wave equation $\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$. Consider the variables $X^+ = x + vt$ and $X^- = x - vt$. After much chain rule, we get $\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial X^+}\frac{\partial X^+}{\partial t} + \frac{\partial\psi}{\partial X^-}\frac{\partial X^-}{\partial t} = v\left(\frac{\partial\psi}{\partial X^+} + \frac{\partial\psi}{\partial X^-}\right)$, and a second derivative. Finally, we get:

$$\frac{\partial^2\psi}{\partial X^+\partial X^-} = 0$$

Hence all solutions look like:

$$\begin{aligned}\psi &= f(X^-) + g(X^+)\\ &= f(x - vt) + g(x + vt)\end{aligned}$$

Hence the general solution to the wave equation is a linear superposition of right moving and left moving waves.

Recall that $\psi = A\sin(kz)\cos(\omega t) = \frac{A}{2}[\sin(kz + \omega t) + \sin(kz - \omega t)]$, hence we can write the standing wave as a superposition of oppositely-oriented travelling waves.

Driving the wave Impose boundary conditions by driving the wave from the left with amplitude $A\cos\omega t$. Then the string has to satisfy: $\psi(0, t) = A\cos\omega t = f(-vt)$, where the g parts vanishes since there are no left-moving waves. We observe that the solution can be written as $\psi(x, t) = A\cos\left[\frac{\omega}{v}(x - vt)\right] = A\cos(kx - \omega t)$.

Phase velocity Write $\omega = v_\phi k$.

Waves in matter Index of refraction $n = \sqrt{\epsilon_r\mu_r}$ so that $v_\phi = \frac{c}{n}$.

Snell's Law Let an incoming wave be written as $\vec{E}_I = \vec{E}_{0,I}\cos(\vec{k}_I \cdot \vec{r} - \omega t)$ and strike the surface at an angle θ from the normal. Let the reflected wave be $\vec{E}_R = \vec{E}_{0,R}\cos(\vec{k}_R \cdot \vec{r} - \omega t)$ and the transmitted wave be $\vec{E}_T = \vec{E}_{0,T}\cos(\vec{k}_T \cdot \vec{r} - \omega t)$. Let $\vec{k}_R$ and $\vec{k}_T$ be θ_R, θ_T from the normal. After plugging in boundary conditions, we have that:

$$\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$$

Hence we have that $k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$. But we have that $\omega = \frac{c}{n_I}k_I = \frac{c}{n_R}k_R = \frac{c}{n_T}k_T$ is the same for all the waves. Since $n_I = n_R$, since they are in the same medium, we have that $\sin\theta_I = \sin\theta_R$ or $\theta_I = \theta_R$. Similarly, we have that $n_T \sin\theta_T = n_I \sin\theta_I$, which is Snell's law.

Impedance Measures the "inertia" of the system. Consider the angle of the string at the driving point $\vec{F}_{vertical} = T\sin\theta \approx T\tan\theta = T\frac{\partial\psi}{\partial z}_{z=0}$. We want Z that satisfies $|F_{vertical}| = Z|\dot{\psi}|$. Noting that power is given by $P = Fv$, blah blah blah

5.3 Thursday 30 Oct 2014

Dispersion relations for all frequencies Recall that:

$$\omega^2 = \begin{cases} \omega_0^2 - 4\omega_1^2\sinh^2(ka/2), & \omega < \omega_0 \\ \omega_0^2 + 4\omega_1^2\sin^2(ka/2), & \omega_0 < \omega < \sqrt{\omega_0^2 + 4\omega_1^2} \\ \omega_0^2 + 4\omega_1^2\cosh^2(ka/2), & \omega > \sqrt{\omega_0^2 + 4\omega_1^2} \end{cases}$$

Frequency dependence of dielectric constant for plasma $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$, where $\omega_p^2 = \frac{4\pi Ne^2}{m_e}$. The plasma frequency for a metal is in the UV light range. Hence visible light is reflected by the metal while ultraviolet light penetrates the metal. Note that when we consider the coupling of the electrons, then $\epsilon(\omega) = 1 + \frac{4\pi Ne^2}{m(\omega_0^2 - \omega^2)}$. Around the resonant frequency, the dielectric constant goes to infinity, then becomes negative as the frequency is increased further. The dielectric constant then increases and returns to unity for large frequencies. But since $n = \sqrt{\epsilon}$, when the dielectric constant is negative, we have that n is also imaginary. Since $\omega = \frac{c}{n}k$, then k is also imaginary and we get evanescent waves.

Refraction Snell's law: $n_1\sin\theta_1 = n_2\sin\theta_2$.

Critical angle When $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 > 1$, no transmission occurs and when $\sin\theta_2 = 1$, we have that $\theta_1 = \sin^{-1}\frac{n_2}{n_1}$ is the critical angle.

Impedance Consider a motor driving a string with tension T. The force of the string on the motor is $T\sin\theta \approx T\tan\theta = T\left.\frac{\partial\psi}{\partial z}\right|_{z=0}$. Hence the force of the motor on the string is $-T\left.\frac{\partial\psi}{\partial z}\right|_{z=0}$. Hence the power that the motor introduces into the string is $P = Fv = -T\left.\frac{\partial\psi}{\partial z}\right|_{z=0}\left.\frac{\partial\psi}{\partial t}\right|_{z=0}$. But we know that the first derivatives are related because $\psi = f(kz - \omega t)$. Hence we have that $\psi' = \frac{-1}{v}\dot{\psi}$. Substituting this into the power equation, we obtain $P = -T(\frac{-1}{v}\dot{\psi}(0))\dot{\psi}(0) = \frac{T}{v}\dot{\psi}(0)^2 = vT(\psi')^2$. Note that $\frac{T}{v}\dot{\psi}(0)^2$ can be related to the kinetic energy (per unit time) and $vT(\psi')^2$ is like a potential energy (per unit time). Since $v = \sqrt{\frac{T}{\mu}}$, we have that $\frac{T}{v} = \sqrt{\mu T}$. We define the quantity Z, the impedance, to be the ratio between the power P and $\dot{\psi}(0)^2$. Then we have that $Z = \sqrt{\mu T}$.

Analogues In the LC transmission line, $T \to (C/a)^{-1}, \mu = (L/a)$, $v_\phi = \frac{a}{\sqrt{LC}}, Z = \sqrt{\frac{L/a}{C/a}}$. Note that the impedance is not the same as the electrical impedance!

Chapter 6

Week 6

6.1 Tuesday 4 Nov 2014

Driving Force on String Recall that $P_{driver}(t) = -T \left.\frac{\partial \psi}{\partial z}\right|_{z=0} \frac{\partial \psi}{\partial t}$. Since we have that $\psi' = \frac{-1}{v_\phi}\dot{\psi}$, we can write $P(t) = vT\left(\frac{\partial \psi}{\partial z}\right)^2 = Z\left(\frac{\partial \psi}{\partial z}\right)^2$, where $Z = \frac{T}{v_\phi} = \sqrt{\mu T}$.

Transmission Line Analogue Recall that $T \to (C/a)^{-1}, \mu \to (L/a)$. Then $Z = \sqrt{\frac{L/a}{C/a}}$.

Propagation of Power We can think of the Power as being an Energy Density multiplied by the Phase Velocity. For instance, for a string, energy per unit wavelength $= \frac{U}{\lambda} = \frac{KE+PE}{\lambda}$. Write $\psi = Ae^{i(kz-\omega t)}$. Then we have that $\frac{U}{\lambda} = \frac{1}{4}\mu\omega^2 A^2 + \frac{1}{4}Tk^2A^2$. Note that the 1/4 comes from the 1/2 coefficient of the KE and PE as well as averaging over the $\cos^2 \omega t$ in 1 period. For a transmission line, we have $\frac{U}{\lambda} = \frac{1}{2}(C/a)Voltage^2 + \frac{1}{2}(L/a)I^2$, in the lumped case. For electromagnetic waves, we have that $\frac{u}{Volume} = \frac{1}{8\pi}\epsilon < E^2 > + \frac{1}{8\pi\mu} < B^2 >$. For sound, we have $\frac{u}{cross-sectionalarea} = \frac{1}{4}\rho V_{sound}\omega^2|A|^2 + \frac{1}{2}Kv_{sound}k^2|A|^2$, where K is the adiabatic compressibility.

Energy Absorption Consider a mass placed at the end of a string with a travelling wave. Then the force acting on the mass is given by $F = T\left(\frac{\partial \psi}{\partial z}\right)_{z=L}$, which we can write as $\frac{-T}{v_\phi}\frac{\partial \psi}{\partial t}$. Note that the force is a damping force since it is proportional to the velocity. Hence the power carried by the string is most efficiently absorbed (maximised) if the load has exactly the right damping constant $Z_{load} = Z_{string}$. Write that the restoring force of the load is given by $F = -Z\frac{\partial \psi}{\partial t}$. This is called impedance matching.

Sound: Energy Absorption Recall that for sound, $K = \gamma P_0$, where K is the adiabatic compressibility: $K = \frac{-\partial V}{V \partial p}$. The impedance of air, which is $Z = \sqrt{\gamma P_0 \rho} = 420 Pa \cdot s/m = 42 dynes \cdot s/cm$.

Sound Intensity Define as the energy per unit time per unit area (power flux) to be $Z\left(\frac{\partial \psi}{\partial t}\right)^2 = \frac{1}{Z}\left[-\gamma P_0 \frac{\partial \psi}{\partial z}\right]^2$. Observe that $-\gamma P_0 \frac{\partial \psi}{\partial z}$ is the force per unit area along the longitudinal direction of the compressed gas acting on the region surrounding it. Call this the gauge pressure. The loudness of a sound refers to the energy deposited onto the eardrum, which is proportional to the gauge pressure squared. Then, the intensity is proportional to the amplitude squared (square law detector). The ear is also insensitive to phase!

Decibels Define $1dB = 10\log_{10}\frac{I}{I_0}$, where I_0 is a reference intensity, which is the intensity for sound to be barely audible $(10^{-12}Wm^{-2})$. Alternatively, we can put this in amplitudes (gauge pressure) to obtain that $1dB = 20\log_{10}\frac{P_g}{P_{g,0}}$. Then $P_{g,0} = 20\mu Pa = 2 \times 10^{-10} atm$.

Reflection Consider two strings with different mass per unit lengths μ_1 and μ_2. Let the tension in each string be equal to T. Then we assume that an incoming wave approaches from infinity $\psi(x,t) = Ae^{i(\omega t - kx)}$. Consider the waves ψ_i, ψ_r, ψ_t, the incident wave, reflected wave and transmitted wave respectively. In the first region (μ_1), we have $\psi_1 = \psi_i + \psi_r = Ae^{(\omega t - k_1 x)} + A_r e^{(\omega t + k_1 x)}$. In the second region, we have $\psi_t = A_t e^{i(\omega t - k_2 x)}$. Now we know that $k_1 = \frac{\omega}{v_1}, v_1 = \sqrt{\frac{T}{\mu_1}}$, and $k_2 = \frac{\omega}{v_2}$, $v_1 = \sqrt{\frac{T}{\mu_2}}$. Imposing boundary conditions: the string has to be continuous and its first derivative in time must be continuous as well. Hence we have that $\omega_1 = \omega_2$. We also require that the force be continuous across the interface. The force is due to the slope of each end multiplied by the tension. The interface should have zero net force because it is massless. Hence we have that $T\left.\frac{\partial \psi_1}{\partial x}\right|_{x=0} = T\left.\frac{\partial \psi_2}{\partial x}\right|_{x=0}$. Hence we have $A + A_r = A_t$ and $k_1(A - A_r) = k_2 A_t$.

Let $Z_1 = \sqrt{T\mu_1}, Z_2 = \sqrt{T\mu_2}$. Then we have $R = \frac{A_r}{A} = \frac{Z_1 - Z_2}{Z_1 + Z_2}, T = \frac{A_t}{A} = \frac{2Z_1}{Z_1+Z_2}$. Note that $1 + R = T$. Note also that since Z_1 and Z_2 are real, there are no phases induced.

6.2 06 Nov 2014 Thursday

Driving a Coaxial Cable At an unterminated end, the impedance is infinity, hence the current will be reflected with negative sign. Recall that the capacitance per unit length (cgs units) is $\frac{C}{a} = \frac{\epsilon}{2}\frac{1}{\ln(r_2/r_1)}$ and the inductance per unit length is $\frac{L}{a} = \frac{2\mu}{c^2}\ln(r_2/r_1)$. The phase velocity is $v_\phi = \frac{c}{\sqrt{\epsilon\mu}} \approx \frac{c}{\sqrt{\epsilon}}$ since $\mu \approx 1$ for most materials. The impedance is $Z = \frac{2}{cn}\ln(r_2/r_1) = \frac{60\Omega}{n}\ln(r_2/r_1)$ where the latter is usually the case for coaxial cables. Consider a function generator with an output impedance of 50Ω and $Z_{LC} = 50\Omega$.

Boundary conditions for coaxial cables For an open circuit (infinity impedance), the current interferes destructively at the end so that the reflected current has negative sign. For a short circuit, the current propagates back with the same sign. However, the voltage does not follow the sign of the current and reflects back with the same sign for the open circuit and with the negative sign for the short circuit. Hence at the end of an open circuit, the voltage is $2V$.

Types of reflection We have amplitude reflection, force reflection and energy reflection. Recall that $R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$. Note that since $Z \propto v_\phi$, we can write it as $R = \frac{v_1 - v_2}{v_1 + v_2}$. Then $T = \frac{2v_1}{v_1+v_2}$. Note that the force on the string is given by $f = T_0\frac{\partial\psi}{\partial z}$, where T_0 is the tension in the string. Let $\psi = Ae^{i(\omega t - kz)}$ Then in the initial medium, we have $f = f_{incident} + f_{ref} = -ik_1AT_0e^{i(\omega t - k_1 z)} + ik_1RAT_0e^{-(\omega t + k_1 z)}$. Hence the reflection coefficient for the force is given by $f_{ref}/f_{incident} = -R$. This is because the sign of k_1 is different for the reflected wave and the incident wave, hence when taking derivatives, we obtain a different reflection coefficient for the force. Note that the transmitted coefficient for the force is still T, because the wave still moves in the positive direction.

Parallels

System	Displacement	Transverse velocity	Force	Impedance
String	ψ	$\dot{\psi}$	$T\frac{\partial\psi}{\partial z}$	$\sqrt{T\mu}$
Transmission Line	Q	I	V	$\sqrt{\frac{L/a}{C/a}}$
EM waves	$\vec{A}$, vector potential	$\vec{B}$	$\lvert\vec{E}\rvert = Z\lvert\vec{B}\rvert$	$Z = \sqrt{\frac{\mu}{\epsilon}} \approx \frac{1}{n}$ usually,CGS

Reflection of EM waves Note that since $n \propto \frac{1}{v}$, we have that $R = \frac{n_2 - n_1}{n_1 + n_2}$ and $T = \frac{2n_2}{n_1+n_2}$.

Impedance of free space From plane-wave solution to Maxwell's equations. By definition, $Z_0 = \frac{E}{H} = \mu_0 c_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c_0} \approx 377\Omega$.

Energies of oscillation Note that the energy can be written as $E = KE + PE$. For a string, write $\psi = A\cos(\omega t - kx)$ and we have $KE = \int \frac{1}{2}(\mu dx)|\dot{\psi}|^2 = \int \frac{1}{2}\mu dx[-A\omega\sin(\omega t - kx)]^2 = \frac{1}{2}\mu A^2\omega^2\int_\lambda \sin^2(\omega t - kx)dx = \frac{1}{4}\mu A^2\omega^2\lambda$. Also, $PE = \frac{1}{2}kx^2 = T_0\int\sqrt{dx^2+dy^2} - dx = T_0\int dx\left[\sqrt{1+\left(\frac{dy}{dx}\right)^2} - 1\right] \approx T_0\int dx\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \frac{1}{2}T_0\int_\lambda\left(\frac{\partial\psi}{\partial x}\right)^2 dx = \frac{1}{2}T\int_\lambda\left[kA\sin(\omega t - kx)\right]^2 dx = \frac{1}{4}k^2A^2T\lambda$. Recall also that for a string, $v_\phi^2 = T/\mu = (\omega/k)^2$ and $\lambda = \frac{2\pi}{k}$. Rewriting both equations, we have that $KE = \frac{1}{2}A^2Tk\pi = PE$ averaged over time and space. This is related to the Virial theorem. Hence we have a total energy that is πA^2Tk.

Reflected energy We have that $E \propto TA^2k$. Hence $R_e = \frac{T_1k_1B^2}{T_1k_1A^2} = \left(\frac{z_1-z_2}{z_1+z_2}\right)^2$ and $T_e = \frac{T_2k_2C^2}{T_1k_1A^2} = \left(\frac{2z_1}{(z_1+z_2)}\right)^2\frac{z_2}{z_1} = \frac{4z_1z_2}{(z_1+z_2)^2}$. Observe that $R_e + T_e = 1$, which makes sense in conservation of energy.

Relations between R and T Note that for amplitudes and velocities (referring to the local oscillation velocity $\frac{\partial\psi}{\partial t}$), we have $1 + R_a = T_a$. For forces, we have $1 = R_f + T_f$. For energy, we have $R_e + T_e = 1$. Note that there is a difference in formula between forces and energy, since R_e, T_e are non-negative (they are the squares of some function), while R_f and T_f can go negative.

Energy in parallel systems Recall that the energy per unit wavelength is $\frac{E}{\lambda} = \frac{1}{4}\mu\omega^2A^2 + \frac{1}{4}Tk^2A^2$. The energy per unit length in the LC lines is $\frac{E}{a} = \frac{1}{2}(C/a)^2V^2 + \frac{1}{2}(L/a)I^2$. For EM waves, we have energy per unit volume $\frac{E}{V} = \frac{1}{8\pi}\epsilon|\vec{E}|^2 + \frac{1}{8\pi\mu}|\vec{B}|^2$.

For sound, we have $\frac{E}{Area} = \frac{1}{4}\rho v\omega^2 A^2 + \frac{1}{4}Kvk^2A^2$. In SI units, $\frac{E}{V} = \frac{1}{2}\epsilon_0|E|^2 + \frac{1}{2\mu_0}|B|^2$.

Multiple boundaries: Antireflection Coating example Consider light incident on glass of $n = 1.5$ from air. Then the intensity of reflection goes as $\left(\frac{1-n}{1+n}\right)^2 \approx 0.04$. We can reduce this by putting a thin-film coating on the glass. Let the coating have thickness l and index of refraction n_1. Let the index of refraction of glass be n_2. We want the reflected wave from n_2 to interfere destructively with the reflected wave from n_1. Note that we can write this as $\frac{I_r}{I_i} = |R_1 + T_{10}e^{ikl}R_2e^{ikl}T_{01}|^2 \approx 0$. The first term refers to the single reflection. The second term refers to the transmission at the first interface, propagation through the coating, reflection at the second interface, propagation to the coating, transmission at the first interface at the opposite direction. Note that T_{01} refers to the amplitude transmission coefficient from medium 0 to 1 (air to coating). T_{10} refers to the amplitude transmission coefficient from medium 1 to 0. R_1 and R_2 are the amplitude reflection coefficients from medium 0 to 1 and 1 to 2 respectively. Note that if $n_1 < n_2$, then R_1 and R_2 will be negative. Hence we need e^{2ikl} to be negative to achieve the destructive interference. Hence we obtain that $2kl = \pi$ or $l = \frac{\lambda}{4}$, we want the thickness of the coating to be a **quarter wavelength** thick. Make the approximation $T_{01} \approx T_{10} \approx 1$. Then we want $R_1 = R_2$ to achieve zero intensity of reflection. Then we have that $n_1 = \sqrt{n_2}$, the coating has index of refraction that is about the square root of the glass index.

Antireflection Coating: Now with more bounces We note that $T_{01} = \frac{2n_1}{1+n_1}$ and $T_{10} = \frac{2\cdot 1}{1+n_1}$. Then $I_r/I_i \approx 10^{-6}$, which is a vast improvement over the 0.04 earlier. Now we consider multiple reflections. For the double reflection case, we have $I_r/I_i = |R_{01} - T_{10}R_{12}T_{01} + T_{10}R_{12}R_{21}R_{12}T_{01}|^2 \approx 10^{-10}$, where we omit the e^{ikl} because we have chosen the quarter wavelength so $e^{2ikl} = -1$.

Fabry-Perot Cavities / Optical Resonant Cavities Recall the Zeeman splitting experiment. We have an incident wave that experiences multiple reflections and transmissions in between two partially transmitting plates. Note that the intensity of the ray that experiences n reflections is given by T^2R^{2n}. Hence we can write $I_t/I_i = |T^2e^{ikl} + T^2R^2e^{3ikl} + \ldots + T^2R^{2n}e^{(2n+1)ikl} + \ldots| = T^4\frac{1}{|1-R^2e^{2ikl}|^2}$ after summing the geometric series. We can write this as $I_t/I_i = \frac{T^4}{1+R^4+2R^2\cos(2kl)}$. Now we note that $R_e + T_e = (R_a)^2 + (T_a)^2 = 1$, we can write $I_t/I_r = \frac{1+R^4-2R^2}{1+R^4-2R^2\cos 2kl}$, which becomes 1 when $2kl = 2\pi n, n \in \mathbb{Z}$. Hence if we choose $l = \frac{\lambda}{2}n$, the transmission coefficient becomes unity and all incident light passes through. But if the distance is slightly displaced from the optimal length, then the transmittance drops very rapidly. Hence if we plot the transmittance against separation for a single incident wavelength, we have very narrow resonance curves centred at $l = \frac{\lambda}{2}n$.

Types of Polarization s-polarization: perpendicular to the plane of incidence. p-polarization: parallel to the plane of incidence (i.e. in the plane of incidence)

Fresnel's Equations (power reflection coefficient)

$$R_s = \left|\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_1 + n_2\cos\theta_t}\right|^2$$

$$R_p = \left|\frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}\right|^2$$

obtain the transmission coefficient by noting that $R + T = 1$ for power coefficients. Note that R_p vanishes at Brewster's angle $\theta_B = \tan^{-1}\frac{n_2}{n_1}$.

Chapter 7

Week 7

7.1 Tuesday 11 Nov 2014

Information encoding AM: $A(t)$, FM: $\omega_0 + \delta\omega$, ϕM: $\phi = \phi_0 + \delta\phi(t)$, last one is phase modulation.

Phase modulation Modulated signal is $A\sin(\omega_c t + m(t) + \phi_c)$. ω_c is the carrier amplitude.

Localized wave superposition Write a general travelling wave as $f(z,t) = \sum_n A_n \cos(\omega_n t - k_n z)$. Let $u = z - v_\phi t$. Then we can write $f(z,t) = \sum_n A_n \cos(k_n u)$.

Example: Beats Recall that $\psi(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t) = 2A\cos(\omega_{mod})\cos(\omega_{av} t)$, where $\omega_{mod} = \frac{\omega_1 - \omega_2}{2}$, $\omega_{av} = \frac{\omega_1 + \omega_2}{2}$. Then the period of modulation can be written as $T_{mod} = \frac{1}{2}\frac{2\pi}{\omega_{mod}}$. To make a pulse, we just make T_{mod} small (i.e. $\omega_1 \approx \omega_2$) and make $T_{periodic} \to \infty$. In this case, we have $T_{mod} = T_{periodic}$ for pure beats. To achieve a pulse, we need to superpose a large number of frequencies. Under this condition, we have:

$$\omega_{periodicity} = \frac{2\pi}{\delta\omega}\frac{1}{2}$$
$$\omega_{mod} = \frac{2\pi}{\Delta\omega}\frac{1}{2}$$

where $\delta\omega$ is the spacing between frequencies and $\Delta\omega$ is the spacing between the highest and lowest frequencies. Note that we can make $\delta\omega$ arbitrarily small by taking a very large number of frequencies. Call $\Delta\omega$ the bandwidth.

Uncertainty relations Note that for a large number of frequencies forming a pulse, we have:

$$\Delta\omega\Delta t$$
$$\Delta k \Delta u, u = z - v_\phi t$$
$$\Delta k \Delta z$$

all of order unity. These are all unitless!

Pulses on a string Consider the case where we have a string of length L along the axis z. Recall that we wrote an arbitrary pulse by assuming that it continued periodically for all positions across infinity. To make a non-periodic pulse, we need to make $L \to \infty$.

Continuous Fourier Transform In the limit where $L \to \infty$, we take $k = \frac{n\pi}{L}$ to be a continuous parameter, and we can let small changes in k be $\delta k = \frac{\pi}{L} dn$. We hence replace the sum across n with an integral over dn, or equivalently, an integral over $\frac{L}{\pi} dk$. Hence we write:

$$f(z) = \int_0^\infty dk \frac{L}{\pi} b_n \cos(kz) + \int_0^\infty dk \frac{L}{\pi} a_n \sin(kz)$$

Call $\frac{Lb_n}{\pi} = b(k)$ and $\frac{La_n}{\pi} = a(k)$. Hence we have:

$$f(z) = \int_0^\infty dk b(k) \cos(kz) + \int_0^\infty dk a(k) \sin(kz)$$

If $f(z)$ is defined for negative z as well, we may also integrate from negative infinity, but we will need to correct the integrals by multiplying by half. Hence we can write the transform generally as:

$$f(z) = \frac{1}{2}\int_{-\infty}^\infty dk b(k) \cos(kz) + \frac{1}{2}\int_{-\infty}^\infty dk a(k) \sin(kz)$$

Recalling the definition of b_n and a_n:

$$b(k) = \frac{1}{\pi}\int_{-\infty}^\infty f(z)\cos(kz)dz$$
$$a(k) = \frac{1}{\pi}\int_{-\infty}^\infty f(z)\sin(kz)dz$$

Making qualitative observations We note that if $f(z)$ is even, then we expect $b(k)$ to be nonzero and $a(k)$ to be zero. Similarly, if $f(z)$ is odd, we expect $b(k) = 0$ and $a(k) \neq 0$.

Time-dependence Consider a pulse $f(z)$ at $t = 0$. Then we perform the Fourier transform and know $b(k)$ and $a(k)$. Then we have the time-dependent function: $\psi(z,t) = \int_{-\infty}^\infty (b(k)\cos ku + a(k)\sin ku)dk$, where $u = z - vt$.

Inverse Laplace Transforms Observe that:

$$b(k) = \frac{1}{\pi}\int_{-\infty}^\infty \psi(u)\cos(ku)du$$
$$a(k) = \frac{1}{\pi}\int_{-\infty}^\infty \psi(u)\cos(ku)du$$

This is because:

$$\int_{-\infty}^\infty \cos(ku)\cos(k'u)du = \frac{1}{2}2\pi\delta(k-k')$$
$$\int_{-\infty}^\infty \sin(ku)\sin(k'u)du = \frac{1}{2}2\pi\delta(k-k')$$

Dirac Delta Functions Characteristics: $\int_{-\infty}^\infty \delta(k-k')dk = 1$, $\int_{-\infty}^\infty \delta(k-k')f(k)dk = f(k')$.

Fourier Transforms

$$\delta(x) = \int_{-\infty}^\infty e^{-2\pi ikx}dk$$
$$\int_{-\infty}^\infty e^{-2\pi ikx}\delta(x)dx = 1$$

7.2 Wednesday 12 Nov 2014 Recitation

Dashpot Consider a dashpot attached to the end of a string. The dashpot introduces a drag force $F_R = -Z_R v(t)$.

Reflection coefficients in terms of wave numbers $R = \frac{k_1-k_2}{k_1+k_2}, T = \frac{2k_1}{k_1+k_2}$. We also have that $\omega = vk, v = \sqrt{T/\rho}, Z = \sqrt{T\rho}$. Hence we can just replace ks with Zs. Note that T is continuous across a boundary but ρ, the mass density, is not. Rearranging, we obtain $k = \frac{\omega}{v} = \omega\sqrt{\frac{\rho}{T}} = \omega\sqrt{\frac{Z^2/T}{T}} = \frac{\omega Z}{T} \implies k \propto Z$.

7.3 Thursday 13 Nov 2014

Review Recall that we can write solutions to the wave equation as $f(u), u = x - vt$. We consider the cosine-like components of this function as the infinite set $\{f_k(u)\} = \{\cos ku\}$, where k is a continuous real variable. Then the condition for orthonormality is fulfilled because $\int_{-\infty}^{\infty} \cos k'u \cos ku du = \pi\delta(k - k')$. Hence we can write the general function of u using these orthonormal functions: $\psi(u) = \int_{-\infty}^{\infty} dk[b(k)\cos(ku) + a(k)\sin(ku)]$ for continuous functions $b(k), a(k)$.

Waves on a ring Consider a ring with one parameter θ. This is a Dirichlet boundary condition because we require that $f(\theta) = f(\theta + 2\pi)$.

Complex Fourier Transform Define $\psi(u) = \int_{-\infty}^{\infty} c(k)e^{iku}dk$. Note that orthonormality becomes $\int_{-\infty}^{\infty} e^{i(k-k')u}du = 2\pi\delta(k - k')$. This comes from the definition of the delta function from the Fourier transform (making the substitution $u = 2\pi k$:

$$\delta(x) = \int_{-\infty}^{\infty} e^{-2\pi ikx}dk = \int_{-\infty}^{\infty} e^{-iux}d(u/2\pi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}du = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}du$$

We now perform this orthonormality condition on the original function:

$$\begin{aligned}
\int_{-\infty}^{\infty} due^{-ik'u}\psi(u) &= \int_{-\infty}^{\infty} due^{-ik'u}\int_{-\infty}^{\infty} c(k)e^{iku}dk \\
&= \int_{-\infty}^{\infty} c(k)dk\int_{-\infty}^{\infty} due^{i(k-k')u}du \\
&= \int_{-\infty}^{\infty} dkc(k)2\pi\delta(k' - k) \\
&= 2\pi c(k')
\end{aligned}$$

Comments on the complex Fourier Transform If the function is even $\psi(u) = \psi(-u)$, then we know that it is going to be cosine-like, and we can just evaluate the cosine-like Fourier transform. We can also perform the complex fourier transform to obtain that $C^*(-k) = C(k)$. Conversely, if $\psi(u) = -\psi(-u)$, then we will have that $C^*(-k) = -C(k)$.

Inverse Fourier Transform We note that:

$$\psi(u) = \int_{-\infty}^{\infty} dkC(k)e^{iku} \iff C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(u)e^{-iku}du$$

Call $C(k)$ the Fourier transform of $\psi(u)$ and $\psi(u)$ the inverse Fourier transform of $C(k)$. Alternatively, we can "split" the 2π factor to rewrite the functions as:

$$\psi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkC(k)e^{iku} \iff C(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(u)e^{-iku}du$$

Examples Consider a pulse that is very well localised in frequency. Let $b(k) = \delta(k - k_0)$, centred at k_0. This is a pure sinusoid, and hence extends to infinity. Computing its Fourier transform, we note that the delta function is even about $k = k_0$. Hence we examine the cosine-like solutions:

$$\begin{aligned}
\psi(u) &= \int_{-\infty}^{\infty} \delta(k - k_0)\cos kudk \\
&= \int_{-\infty}^{\infty} \delta(k)\cos((k + k_0)u)dk \\
&= \cos(k_0u) \\
&= \cos(k_0x - \omega t)
\end{aligned}$$

hence we note that the Fourier transform of a delta function in k-space is a plane wave.

Box function Note that the delta function can be thought of as the limiting case of a box-function with the width going to zero and the height going to infinity. Then we write $b(k) = \alpha\left[\theta(k-(k_0-\Delta k)) - \theta(k-(k_0+\Delta k))\right] = \begin{cases} 0, & k < k_0 - \Delta k \\ \alpha, & k_0 - \Delta k < k < k_0 + \Delta k \\ 0, & k > k_0 + \Delta k \end{cases}$ where θ represents the Heaviside function (step function). Then we have the Fourier transform:

$$\begin{aligned} \psi(u) &= \int_{k_0-\Delta k}^{k_0+\Delta k} \alpha\cos(ku)dk \\ &= \frac{\alpha}{u}\left[\sin[(k_0+\Delta k)u] - \sin[(k_0-\Delta k)u]\right] \\ &= 2\alpha\Delta k\frac{\sin(\Delta k u)}{\Delta k u}\cos(k_0 u) \\ &= 2\alpha\Delta k\ \mathrm{sinc}(\Delta k u)\cos(k_0 u) \end{aligned}$$

Note that for sinc $= \frac{\sin x}{x}$, we have that $\lim_{a\to 0}\frac{\mathrm{sinc}(x/a)}{a\pi} = \delta(x)$, a delta function. Hence the limiting case to the box function where $\Delta k \to 0$ will be a delta function.

Sinc function Note that we can characterise the sinc function by considering the full-width at half maximum. For the function $\mathrm{sinc}(\Delta k u)$ as a function of u, we have that the FWHM is $2\Delta u = \frac{\pi}{\Delta k}$. So we have that $\Delta u\Delta k = \frac{\pi}{2}$.

Gaussian function The Fourier Transform of a Gaussian is a Gaussian. The Gaussian is a minimal uncertainty function because it gives a minimal $\Delta t\Delta\omega = \frac{1}{2}$. In comparison, the box function gives an uncertainty relation of $\Delta t\Delta\omega = \frac{\pi}{2}$. The normalised Gaussian looks like:

$$\psi(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-x_0)^2/2\sigma^2}$$

Moments Define the nth moment of the distribution $\psi(x)$ about the point $x = 0$ to be $\int x^n\psi(x)dx$. The zero-th moment is just the integral, which can be computed to be $\int\psi(x)dx = 1$. The RMS about the mean of a Gaussian is the standard deviation of the Gaussian.

FT of a Gaussian Given $b(k) = \frac{1}{\sqrt{2\pi\sigma_k^2}}e^{-(k-k_0)^2/2\sigma_k^2}$, we have that:

$$\psi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b(k)e^{iku}dk = \frac{1}{\sqrt{2\pi/\sigma_k^2}}e^{-u^2\sigma_k^2/2}e^{ik_0u}$$

Note that $\frac{1}{\sqrt{2\pi/\sigma_k^2}}e^{-u^2\sigma_k^2/2}$ is the slowly-varying Gaussian amplitude envelope ande^{ik_0u} refers to the rapidly oscillating terms within the envelope. We can define $\sigma_u = \frac{1}{\sigma_k}$ such that $\sigma_u\sigma_k = 1$. Then we can write the Gaussian envelope as $\frac{1}{\sqrt{2\pi\sigma_u^2}}e^{-u^2/2\sigma_u^2}$. We note that we can obtain the usual uncertainty relation by examining the intensity, which goes as the square of $\psi(u)$. Hence we note that the $[b(k)]^2$ goes as $e^{-(k-k_0)^2/\sigma_k^2}$ and $[\psi(u)]^2$ goes as $e^{-u^2\sigma_k^2}$ and hence we obtain $\Delta k = \frac{\sigma_k}{\sqrt{2}}$ and $\Delta u = \frac{1}{\sigma_k\sqrt{2}}$ so that $\Delta k\Delta u = \frac{1}{2}$ as expected.

Gaussian Tricks

1. Computing normalisation using polar coordinates.
2. Computing moments: Recall that the Gaussian is even about the mean, hence the odd moments vanish. Also define $I(a) = \int x^n e^{-x^2a}dx$. Then note that we can calculate the even moments by taking the derivative of $I(a)$ with respect to a.
3. Evaluating $e^{-x^2a-xb-c}$ by completing the square.

Chapter 8

Week 8

8.1 18 Nov 2014 Tuesday

Review The Fourier transform of a delta function is a pure sine wave of single frequency. Also, the inverse Fourier transform of a delta function is also a pure sine wave. Hence a delta function in time is a sine wave in frequency.

Ringdown A ring down has time-domain representation $\psi(t) = \cos(\omega_0 t)e^{-\Gamma t/2}$, for $t > 0$. We introduce the factor of 2 in the exponent so that the intensity goes as $e^{-\Gamma t}$. We write this as $\psi(t) \int_{-\infty}^{\infty} C(\omega)e^{i\omega t}d\omega$ so we have $C(\omega) = \frac{1}{2\pi}\int_0^{\infty} dt\psi(t)e^{i\omega t}$. Transforming this to the complex domain, we write $\psi(t) = Ae^{i\omega_0 t}e^{-\Gamma t/2}$ so:

$$C(\omega) = \frac{A}{2\pi}\int_0^{\infty} dte^{-[i(\omega-\omega_0)+\Gamma/2]t} = \frac{A}{2\pi}\frac{1}{\Gamma/2 + i(\omega - \omega_0)}$$
$$\implies |C(\omega)|^2 = \frac{1}{(\omega-\omega_0)^2 + \Gamma^2/4}$$

this shape is a Lorentzian with a FWHM of Γ for $|C(\omega)|^2$. Hence the impulse response of a harmonic oscillator is a ring-down in the time domain and a Lorentzian in the frequency domain.

Propagation of pulses Write:

$$\psi(x,t) = \int b(k)e^{i(kx-\omega t)}dk, b(k) = \tilde{\psi}(x,0)$$

where we use the tilde to represent the Fourier transform. Assume $b(k)$ is peaked at k_0 with a characteristic width $\Delta k << k_0$.

Note that $\psi(x,0) = \int b(k)e^{ikx}dk$ so we obtain $b(k)$ by taking the Fourier transform of $\psi(x,0)$.

Assume that $b(k)$ is about constant on $k_0 - \Delta k$ to $k_0 + \Delta k$ and is equal to around α. Then we need to represent $\omega(k)$ around k_0. Take the Taylor expansion: $\omega(k) = \omega(k_0) + \frac{d\omega}{dk}|_{k=k_0}(k-k_0) + \ldots$. Then we have:

$$\psi(x,t) = \alpha\int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(kx-\omega_0 t - \frac{d\omega}{dk}(k-k_0)t)}dk$$
$$= \alpha e^{-i\omega_0 t}e^{\frac{d\omega}{dk}k_0 t}\int_{k_0-\Delta k}^{k_0+\Delta k} e^{ik(x-\frac{d\omega}{dk})t}dk$$

Define $u = x - \frac{d\omega}{dk}|_{k=k_0}t$. We also know that the integral of the exponential is going to give a sinc function, hence we have:

$$\begin{aligned}\psi(x,t) &= \alpha e^{-i\omega_0 t} e^{i\frac{d\omega}{dk}k_0 t} \left.\frac{e^{iku}}{iu}\right|_{k_0-\Delta k}^{k_0+\Delta k} \\ &= 2\alpha e^{-i\omega_0 t} e^{ik_0 x} \Delta k \frac{\sin(\Delta k u)}{\Delta k u} \\ &= 2\alpha \Delta k \frac{\sin(\Delta k(x - \frac{d\omega}{dk}t))}{\Delta k(x-\frac{d\omega}{dk}t)} e^{i(k_0 x - \omega_0 t)}\end{aligned}$$

Observe that this wave translates at velocity $\left.\frac{d\omega}{dk}\right|_{k=k_0} = v_g$. Note that if the pulse is not narrow, then the higher order terms in the Taylor expansion of $\omega(k)$ will be relevant, and the wave will disperse in time.

Method of Stationary Phase Given $\psi(u) = \int c(u)e^{iku}dk$, where $u = x - v_\phi t$ and $c(k)$ slowly varying with a peak at k_0. Note that for $\psi(u)$ to be non-zero (constructive interference), then it means that the phase ku must be approximately constant, so that the integral over all k does not cancel the overall value. This occurs when $\frac{\partial}{\partial k}(ku) \approx 0 \implies \frac{\partial}{\partial k}(kx - \omega t) \approx 0 \implies x - \left.\frac{d\omega}{dk}\right|_{k=k_0} t = 0$.

Dispersion in a Plasma Recall that $v_\phi = \frac{c}{\sqrt{\epsilon}} = \frac{\omega}{k}$. Also, we have the dispersion relation $\omega^2 = \omega_p^2 + c^2k^2$. Hence we have $\omega^2 = \frac{c^2k^2}{\epsilon} = \frac{c^2k^2}{1-\frac{\omega_p^2}{\omega^2}}$. Hence we have $v_\phi = \frac{\omega}{k} = c\sqrt{1+\frac{\omega_p^2}{c^2k^2}} > c$. However, the group velocity is $\frac{c}{\sqrt{1+\frac{\omega_p^2}{c^2k^2}}} < c$.

Review of Multivariable Calculus

- Note that $\Box^2 = O_x + O_y + O_z + O_t$, where O_j represents an operator on the jth coordinate. Then solutions to $\Box^2\phi = 0$ can be written by separation of variables as $\psi(x,y,z,t) = \phi_x(x)\phi_y(y)\phi_z(z)\phi_t(t)$. In general, if we can write the operator as a sum of operators on orthogonal co-ordinates, then the solution can be written in separation of variables.
- However, if the operator is not just a simple sum of the operators in different coordinates, then we need to write $\psi(\vec{r},t) = \sum_{ijkl} \phi_{xi}(x)\phi_{yj}(y)\phi_{zk}(z)\phi_{tl}(t)$.
-

3D waves For a single plane-wave in 3D, we write $\psi(\vec{r},t) = e^{ik_x x}e^{ik_y y}e^{ik_z z}e^{-i\omega t} = e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ and $\omega^2 = v^2(k_x^2+k_y^2+k_z^2) = v^2|\vec{k}|^2 = v^2\left(\frac{2\pi}{\lambda}\right)^2$.

Standing Wave in 3D We write $\psi(\vec{r},t) = \sin(k_x x + \delta_x)\sin(k_y y + \delta_y)\sin(k_z z + \delta_z)\cos(\omega t + \phi)$ for a general standing wave. Note that we can engineer a traveling wave in one direction like $\psi(\vec{r},t) = \sin(k_x x)\sin(k_y y)\cos(k_z z - \omega t + \phi)$.

Evanescent waves If $k_j^2 < 0$, then write $\kappa_j = \sqrt{-k_j^2}$ and then in that co-ordinate we have $e^{-\kappa_j j}$, an exponentially decaying wave.

8.2 Recitation 19 Nov 2014

Group velocity $\frac{\omega}{k} = v_\phi \implies \frac{d\omega}{dk} = v_g = v_\phi + k\frac{dv_\phi}{dk}$.

Surface Water Waves (Crawford 6.19) Dispersion relation for ocean waves: $\omega^2 = gk + \frac{T}{\rho}k^3$. T is the surface tension and ρ is the mass density of the water. Group velocity and phase velocity are equal when $g = \frac{T}{\rho}k^2$.

8.3 20 Nov 2014

Isotropic medium Frequency does not depend on the direction of the wave vector, but only on the magnitude $\omega = \omega(|\vec{k}|)$.

EM wave waveguide Consider a rectangular cross section extended in the z direction, dimensions L_x, L_y and infinite in the z direction. We want standing waves in x and y and propagating waves in z. Assume isotropic and linear dispersion relation: $\omega = \frac{c}{n}|\vec{k}|$.

Modes:

- Transverse Electric TE_{nm}, n = number of standing waves (mode) in the x direction, m is the number of standing waves (mode) in the y direction.

- TM
- TEM

Transverse Electric Boundary conditions: Let $\vec{E}$ point in the x direction.$E_x(y=0) = E_x(y=L_y) = 0 \implies E_x(\vec{r},t) = E_0 \sin(k_y y)\cos(k_z z - \omega t + \phi), k_y = \frac{n\pi}{L_y}$. This can be rewritten as $E_x = \frac{E_0}{2}\left(\sin(k_y + k_z z - \omega t) - \sin(-k_y y + k_z z - \omega t)\right)$. Note that the wave can be written as the superposition of two travelling waves with wave vectors $\vec{k}_1 = k_y\hat{y} + k_z\hat{z}$ and $\vec{k}_2 = -k_y\hat{y} + k_z\hat{z}$. Each time the wave hits the boundary, it undergoes specular reflection. Then the dispersion relation looks like $\frac{c}{n}\sqrt{\left(\frac{n_y\pi}{L_y}\right)^2 + k_z^2}$. Note that the k_z term inside the square root has to be real for propagation along z. Observe that we can write the dispersion relation as $\omega = \sqrt{\omega_p^2 + (ck)^2}$, where $\omega_{min} = \frac{c}{n}\frac{\pi}{L_y}$.

Velocities Define the distance between bounces ct. the wave propagates a distance $ct\cos\theta$ in the $\hat{z}$ direction in between bounces. Also note that the wavefront propagates a distance $\frac{ct}{\cos\theta}$ along the z direction in between bounces. Note that the angle can be written as $\cos\theta = \frac{k_z}{\sqrt{k_y^2+k_z^2}}$. Hence the phase velocity, the velocity of the wavefront, appears to exceed the speed of light. But the group velocity, the actual velocity of the transfer of energy, travels slower than light.

Spherical waves Write the wave in spherical coordinates: $\psi(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Far away from the source, the curvature is not noticeable and the wave behaves like a plane wave. Consider a pulse of width Δr propagating at the group velocity. Consider a detector of width dl perpendicular to the direction of propagation. Then the detector captures $\frac{dl}{2\pi r}$ of the energy in 2D. Then the energy $|\psi|^2$ is proportional to $\frac{1}{r}$ so the magnitude of the wave falls of like $\frac{1}{\sqrt{r}}$ in 2D. in 3D, the amplitude falls of like $\frac{1}{r}$.

Solid angle Define the elemental area $d\vec{A} = r^2 d\Omega\hat{r}$, where $d\Omega$ is the solid angle subtended by the detector. In spherical coordinates $d\Omega = \sin\theta d\theta d\phi$. Total solid angle of a sphere is $\iint \sin\theta d\theta d\phi = 4\pi$.

Operators in Polar Coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Operators in Spherical Coordinates (Radial only)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \dots$$

the wave equation has solutions $\psi = \frac{e^{i(kr-\omega t)}}{r}$.

EM waves Note that in vacuum $|\vec{E}| = |\vec{B}|$, cgs units. Also, the energy density of an EM wave is given by $\frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) = \frac{1}{4\pi}|\vec{E}|^2$. The energy flux (energy per area per unit time) is given by the Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$, cgs units.

Radiation from an accelerated point charge Let an electron be accelerated with acceleration a for a time Δt so that it achieves a final velocity $a\Delta t << c$. Consider $r >> \frac{1}{2}a\Delta t^2$. Acceleration of the charge produces traverse waves that carry energy and information away from the source. Note also that no energy is radiated along the axis of acceleration. For an oscillating dipole, the maximum energy transmission occurs perpendicular to the axis of oscillation. Far from the source, only the transverse components survive, and will become a plane wave.

Chapter 9

Week 9

9.1 Tuesday 25 Nov 2014

Radiation from an accelerated point charge Let an electron be accelerated for a short time to achieve velocity $v = a\Delta t$, cruise at v for a time t, then get decelerated back to rest in time Δ. Let the total displacement of the electron be vt. Let the angle between the vertical axis and the observer be θ. Let the electron move in the vertical direction. Then the length of the kink perpendicular to the radial line (i.e. $\hat{\theta}$ direction) is $vt\sin\theta$. We note that the length of the kink parallel to the radial line is $c\Delta$, the thickness of a spherical wheel carrying a perpendicular $E_\perp$.

The initial electric field far away from the electron (no information about acceleration) is $\frac{q}{r^2}$ in CGS units.

Now zoom in on the kink. Draw a Gaussian pillbox perpendicular to the radial direction with one face in the kink and the other outside the spherical shell (electric field there=$\frac{q}{r^2}$). Now the pillbox encloses no charge, and hence the net electric flux is zero through the pillbox. Let the pillbox be small enough such that on the sides of the box, the electric flux cancels. Comparing the electric flux into the face of the pillbox, we obtain that $E_\parallel = E_{outside}$. We also know that $\frac{E_\perp}{E_\parallel} = \frac{vt\sin\theta}{c\Delta t}$. But we know that $E_\parallel = E_{outside} = \frac{q}{r^2}$. Hence we have that $E_\perp = \frac{vt\sin\theta}{c\Delta t}\frac{q}{r^2}$.

Now we note that the observed acceleration of the electron at the position r is the acceleration of the electron at the retarded time $t' = t - \frac{r}{c}$. Let $t = 0$ represent the start of the acceleration. Then $t' = -\frac{r}{c}$ and the observed velocity is $\frac{-ar}{c}$. Note that the component of the electric field due to the acceleration is opposite in direction to a. Then we have the radiation field formula:

$$E_\perp = \frac{-qa(t')\sin\theta}{rc^2}$$

where q is the charge of the particle, $a(t')$ is the acceleration of the charge at the retarded time, θ is the angle between the acceleration and the line of sight, r is the separation between the charge and the observer. Note that the radiation field falls off as $\frac{1}{r}$. Note that along the acceleration axis, no electric field is observed.

Power radiated from an accelerating charge Recall that the Poynting vector is $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$ in CGS units. We know that in vacuum, $|\vec{E}| = |\vec{B}|$ and the direction of the Poynting vector will be in the direction of radiation. Hence $\vec{S} = \frac{c}{4\pi}|\vec{E}|^2\hat{r}$. Explicitly,

$$S = \frac{q^2a^2(t')\sin^2\theta}{4\pi r^2c^3}$$

Recall that $4\pi r^2$ is the surface area of a sphere. To calculate the total power radiated, we need to integrate over all the angles:

$$P = \iint \frac{q^2a^2(t')\sin^2\theta}{4\pi r^2c^3}dA$$

$$P = \int_0^{2\pi}\int_0^{\pi} \frac{q^2a^2(t')\sin^2\theta}{4\pi r^2c^3}r^2\sin\theta drd\theta d\phi = \frac{2q^2a^2(t')}{3c^3}$$

Generalisation Consider acceleration at different times with width Δt. Then we have:

$$dP = \frac{q^2}{c^3}[a(t')]^2 \sin\theta \frac{d\Omega}{4\pi}$$

Rayleigh scattering Consider an electron accelerated by an incident sinusoidal electric field. Then it will oscillate with $x(t) = x_0 \cos \omega t$ with some driving frequency ω. The acceleration then goes as $a(t') = -\omega^2 x_0 \cos(\omega t')$. Then the power radiated becomes:

$$dP = \frac{q^2}{c^3}\omega^4 < x^2(t') > \sin^2\theta \frac{d\Omega}{4\pi}$$

where the angular brackets represent the average over a period. Note that the integral of $\sin^2\theta \frac{d\Omega}{4\pi}$ integrates to $\frac{2}{3}$ over a sphere.

Bound electron If the electron is loosely bound to its ion:

$$m\ddot{x}_e = qE_{inc}(T) - m\omega_0^2 x(t) = -m\omega^2 x(t)$$

where ω is the angular frequency of the driving force. Then we know the amplitude of the response depending on the driving frequency:

$$x(t) = \frac{qE_{inc}(t)}{m(\omega^2 - \omega^2)}$$

now the restoring force for Coulombic forces in a typical molecule is very high $m_e\omega_0^2 \approx \frac{e^2}{a_0^2}$ where $a_0 \approx 10^{-10}m$. Hence we have $\lambda_0 = \frac{2\pi c}{\omega_0} \approx 1200A$, which is in the UV. The incident sunlight has a longer wavelength of around $4000 - 7000A$. Hence we can approximate $\omega_0^2 - \omega^2 \approx \omega_0^2$. Then we have the approximate equation of motion:

$$x(t) = \frac{qE_{inc}(t)}{m\omega_0^2}$$

Now since $\omega^2 x$ is the driving acceleration, the power radiated is $\frac{2}{3}\frac{q^2}{c^3}\omega^4 \frac{q^2 < E_{inc}^2 >}{m^2\omega_0^4}$. Note that higher frequency light is scattered much more (more power for the same incident power) than lower frequency light.

Rayleigh scattering is valid for $\lambda_{inc} >> \lambda_{restoring}$.

Reflection Coefficients in 2D and 3D Note that in 1D, the reflection coefficient was unitless. In 2D, the reflection coefficient has dimensions of length, and in 3D, the reflection coefficient has dimensions of area (the cross-section).

Scattering cross section Consider an incident plane wave with $\vec{k}$. Consider a target with cross section σ. Then the power scattered is equal to the incident flux (i.e. power per unit area) multiplied by a cross section σ. In the case of light with frequency ω, we have that $< \vec{S} >= \frac{c}{4\pi} < \vec{E} \times \vec{B} >$. Then the cross section is defined by:

$$\sigma = \frac{P_{scattered}}{< \vec{S}_{incident} >} = \frac{2}{3}\frac{q^4}{c^3}\frac{\omega^4 < E_{inc}^2 >}{m(\omega_0^2 - \omega^2)^2}\frac{1}{(c/4\pi) < E_{inc}^2 >} = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$$

since energy goes as $\frac{q^2}{r}$ in CGS units, we define $r_e = \frac{q^2}{mc^2}$, which has units of length. Call this the classical electron radius. Then we can write:

$$\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$$

Differential cross section Define:

$$\frac{d\sigma(\omega)}{d\Omega} = \frac{dP/d\Omega}{< S_{inc} >}$$

9.2 26 Nov 2014, Recitation

Higher dimension wave equation $\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi$.

9.3 Useful Trigo Formulae

- $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
- $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
- $\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$
- $\cos a + \cos b = 2 \cos \frac{a-b}{2} \cos \frac{a+b}{2}$
- $\sin(x + \pi/2) = \cos x$
- $\sin(x - \pi/2) = -\cos x$
- $\sin(x \pm \pi) = -\sin x$
- $\cos(x + \pi/2) = -\sin x$
- $\cos(x - \pi/2) = \sin x$
- $\cos(x \pm \pi) = -\cos x$
- $\tan(x + \pi) = \tan x$
- $\tan(x \pm \pi/2) = -\cot x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sin 3x = 3\sin x - 4\sin^3 x$

Chapter 10

Week 10

10.1 Tuesday 2 Dec 2014

Radiation Power Recall that for an accelerating charge, the power is given by $P(r, \theta, t) = \frac{q^2}{c^3}[a(t - \frac{r}{c})]^2 \sin^2\theta \frac{dA}{4\pi r^2}$. The integral over all angles is given by $P(r, t) = \frac{2q^2}{3c^3}[a(t - \frac{r}{c})]^2$. Note that $\frac{r^2 P(r,\theta,t)}{dA}$ is the power per unit area, or power per unit solid angle because the solid angle obeys $dA = r^2 d\Omega$.

Driving force from electric field Let an electric field $\vec{E}$ accelerate the charge (bound with natural frequency ω_0) with frequency ω. Far away from resonances, we can write:

$$x(t) = \frac{qE(t)}{m(\omega_0^2 - \omega^2)}$$

Taking the second partial derivative with respect to time (to get the acceleration to plug into the accelerated charge power formula), we obtain that:

$$< P(r, t) >= \frac{2q^2}{3c^3}\omega^4 \frac{q^2}{m^2(\omega_0^2 - \omega^2)^2} < E(t)^2 >$$

If $\omega_0 >> \omega$, then we have that $P \sim \omega^4 \sim \frac{1}{\lambda^4}$, which is the Rayleigh blue sky law.

Flux using Poynting vector Recall that:

$$< \vec{S} >= \frac{c}{4\pi} < E(t)^2 >= \frac{c}{4\pi}\frac{|E|^2}{2}$$

Scattering cross section Take the ratio between the scattered power and the incident power (i.e. Poynting vector average magnitude). We get:

$$\sigma = \frac{P_{scat}}{< \vec{S} >} = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$$

Recall that we can write the classical electron radius as $r_e = \frac{e^2}{mc^2}$ so that the scattering cross section of the electron is $\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$.

Multiple scatterers Consider a macroscopic target of scatterers, each with cross section σ. Consider, for instance, a solid structure. Then the number of targets per volume is $N = N_A \frac{\rho}{A}$, where A is the number of nucleons per atom. Consider a very thin slice of the material. Then the fraction of the cross section that is filled by scatterers is given by number of scatters per volume times area per scatterer (i.e. cross section) times the thickness. This can be written as :

$$f = N\sigma\Delta z$$

The probability of scattering for a thick block is given by the difference equations:

$$P(transmit, z + \Delta z) = P(transmit, z)(1 - f(z))$$
$$P(z) + \frac{dP}{dz}\Delta z = P(z) - P(z)f(z)$$
$$\frac{dP}{dz} = -P(z)f(z)\frac{1}{\Delta z}$$
$$\implies P(z) = P_0 e^{-N\sigma z}$$

Multiple scattering Consider the differential cross section $\frac{d\sigma}{d\Omega}$. We write:

$$\frac{dN_{scattered,\gamma}}{d\Omega} = Flux_\gamma dt \times \frac{N_A}{A}\rho\Delta z \frac{d\sigma}{d\Omega}$$

in which the scattering is dependent on the angle from the incident γ ray. Note that $\frac{N_A}{A}\rho\delta z$ refers to the number of scattering centers in a slab of thickness δz small.

Two Slit Experiment Consider two slits separated by d, illuminating a wall a distance $L >> d$ away from the slits. Let both slits be illuminated by a single light source placed far away in the negative horizontal direction. Consider the position x_1 on the wall making an angle θ with the horizontal passing through the centre of the slits. Choose L such that $r_1 - r_2$, the difference in distances of x_1 to each of the two slits, is on the order of λ. Consider the sum of the electric field at x_1:

$$E(x_1) = E\cos(\omega t - kr_1 - \phi_1) + E\cos(\omega t - kr_2 - \phi_2) = 2E\cos\left(\omega t - k\frac{r_1 + r_2}{2} - \frac{\phi_1 + \phi_2}{2}\right)\cos\left(k\frac{r_2 - r_1}{2} + \frac{\phi_2 - \phi_1}{2}\right)$$
$$\implies E(x_1) \approx 2E\cos\left(\omega t - k\bar{r} - \bar{\phi}\right)\cos\left(\frac{k\Delta r}{2} + \frac{\Delta\phi}{2}\right)$$

We note that we can approximate $r_2 - r_1 = d\sin\theta = \Delta r$. Then we can write:

$$E(x_1) = 2E\cos(\omega t - k\bar{r} - \bar{\phi})\cos\left(\frac{kd\sin\theta + \Delta\phi}{2}\right)$$

Now note that the cosine term with the time dependence averages to $\frac{1}{2}$ over one cycle. Also take $\phi_1 = \phi_2$. Also note that the intensity goes as the average of the electric field squared over one cycle. Hence we can write:

$$I(x_1) = I_0\cos^2\left(\frac{kd\sin\theta}{2}\right)$$

But note that $x_1 = L\tan\theta \implies x_1 \approx L\theta$. Hence we can write:

$$I(\theta) = I_0\cos^2\left(\frac{kd\sin\theta}{2}\right)$$

Intensity away from the axis Note that we can write $r^2 = L^2 + x^2 = L^2 + L^2\tan^2\theta \implies I(\theta) \sim \frac{1}{r^2} \sim \frac{1}{L^2(1+\tan^2\theta)} = \frac{\cos^2\theta}{L^2}$. Hence the intensity falls away slowly as theta increases in either direction.

Separation between maxima Note that for maxima, require that $\frac{kd\sin\theta}{2} = n\pi \implies d\sin\theta = n\lambda, n \in \mathbb{Z}$.

Coherence Note that a physical source will have a finite frequency width. To achieve a measurement of interference, we require that $\Delta\omega\Delta t_{measurement} << \pi$ so that the time of measurement is short enough so that the measurement can be made within the coherence time of the source.

Hanbury, Brown, Tuiss experiment Measured light from Sirius by modelling Sirius as two sources separated spatially. Used very short measurement times ($10^{-8}s$) and measured the coincidence rate of photons in two detectors separated by several meters. Used the coincidence data to estimate the angular diameter of Sirius.

10.2 Wednesday 3 Nov Recitation

Transverse condition No longitudinal polarization $\hat{n} \cdot \hat{z} = 0$. Polarization is always orthogonal to propagation direction.

Circular Polarization Righthanded:

$$\psi(z,t) = Ae^{i(kz-\omega t)}\hat{x} - Ae^{-i(kz-\omega t)}i\hat{y}$$
$$\psi(z,t) = Ae^{i(kz-\omega t)}\hat{x} + Ae^{-i(kz-\omega t)}i\hat{y}$$

10.3 4 Dec 2014

Spherical waves from a narrow slit $E(r) = \frac{A}{r}e^{ikr}e^{-i\omega t}$.

Intensity from two narrow slits $I(x = L\tan\theta) = c\frac{(2A)^2}{4\pi}\frac{1}{2}\cos^2(\frac{kd\sin\theta}{2} + \Delta\phi)$. Note that the peaks are separated by Δx such that $\frac{kd\sin\theta}{2} = n\pi$ or $d\sin\theta = n\lambda$. Hence the angular separation between maxima is $\frac{\lambda}{d}$.

Wide Slit Consider a wide slit of width D, and split it into a large number N of narrow slits with width d (i.e. $D = (N-1)d$), each producing spherical waves. The electric field due to the narrow slits can be written as:

$$E = A_0e^{-i\omega t}e^{ikr}[1 + e^{i\delta} + e^{2i\delta} + \ldots + e^{Ni\delta}]$$

where $\delta = kd\sin\theta$, the phase difference between two consecutive slits. Note that this is a geometric series, and hence we can write:

$$E = A_0e^{-i\omega t}e^{ikr}\frac{1-e^{(N+1)i\delta}}{1-e^{i\delta}} = A_0e^{-i\omega t}e^{ikr}e^{\frac{(N+1)i\delta}{2}}e^{-i\delta/2}\frac{e^{-\frac{(N+1)i\delta}{2}} - e^{\frac{(N+1)i\delta}{2}}}{e^{-i\delta/2} - e^{i\delta/2}} = A_0e^{-i\omega t}e^{ikr}e^{Ni\delta/2}\frac{\sin(N+1)\delta/2}{\sin\delta/2}$$

We let $N \to \infty$ and replace $A_0N = A$, which is finite. Then the intensity is $I = |A|^2[sinc(t)]^2$ where $t = \frac{kD\sin\theta}{2}$.

Frauenhofer diffraction pattern Minima occur at $\frac{kD\sin\theta}{2} = n\pi, D\sin\theta = n\lambda$. Hence the width of the peak is of order $\frac{\lambda}{D}$. Maxima occur at $D\sin\theta = (n+\frac{1}{2})\lambda$.

Two wide slits Envelope is a "primary peak" that goes as $sinc^2(\frac{kD\sin\theta}{2})$ and the fast oscillation is a "secondary peak" cosine-like $\cos^2(\frac{kd\sin\theta}{2})$. The minima of the fast oscillation occurs as $\frac{Nkd\sin\theta}{2} = n\pi$. The maxima of the slow envelope occurs as $\frac{kd\sin\theta}{2} = n\pi$.

Maximum number of peaks Observe that $\sin\theta \leq 1$. Hence the maximum number of principal peaks is $\frac{D}{\lambda}$.

General Fourier Transform of Slit Note that it can be shown that the diffraction pattern in the far field (in terms of θ) can be written as the Fourier transform of the slit position (in terms of x). Write $B(k_x) = \int f(x)\cos(k_x x)dx$, where $f(x)$ represents the wave function at the slit.

Resolving power of diffraction grating Note that the separation between secondary maxima is of order $\frac{\lambda}{Nd}$ and hence the resolving power goes as $\frac{1}{N}$. Hence differences in wavelength can be resolved to 1 part in N.