Ma2 Book Notes

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Fundamental Theorem of Calculus Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and for $a \leq x \leq b$ defined $G(x) = \int_a^x f(\tilde{x})d\tilde{x}$. Then $\frac{dG}{dx} = f(x)$ and $\int_a^b f(x)dx = F(b) - F(a)$ for any anti-derivative F of f ($F' = f$).

Definition: Solution Given an open interval I that contains t_0, a solution of the initial value problem $\frac{dx}{dt}(t) = f(x, t)$ with $x(t_0) = x_0$ on I is a **continuous** function $x(t)$ defined on I with $x(t_0) = x_0$ and $\dot{x}(t) = f(x, t)$ for all $t \in I$.

Existence and Uniqueness (1st order) If $f(x, t)$ and $\frac{\partial f}{\partial x}(x, t)$ are continuous for $a < x < b$ and for $c < t < d$ then for any $x_0 \in (a, b)$ and $t_0 \in (c, d)$ the initial value problem has a unique solution on some open interval I containing t_0.

Definition: Stable stationary point A stationary point x^* is stable if for all $\epsilon > 0$, $\exists \delta > 0$ such that $|x_0 - x^*| < \delta \implies |x(t) - x^*| < \epsilon, \forall t \geq 0$. A stationary point is unstable if it is not stable.

Definition: Attracting point A stationary point x^* is attracting if $\forall \epsilon > 0, \exists \delta > 0$ such that $|x_0 - x^*| < \delta \implies x(t) \to x^*, t \to \infty$. Attracting points are stable.

Separation of variables $\frac{dx}{dt} = f(x)g(t)$. Check for $f(x_0) = 0$, identify solution in that case. Then divide and solve. Don't forget the absolute values in the ln!!!

Exact Equations The differential equation $f(x, y) + g(x, y)\frac{dy}{dx} = 0$ is exact iff $f_y = g_x$.

Homogeneous equations A first order differential equation is homogenous if it can be written as $\frac{dy}{dx} = F(y/x)$. Make the substitution $u = y/x$, $\frac{dy}{dx} = u + x\frac{du}{dx}$ so that $x\frac{du}{dx} = F(u) - u$, which is separable.

Bernoulli equations The differential equation $\frac{dy}{dx} + p(x)y = q(x)y^n, n \neq 0, 1$ can be solved with the substitution $u = y^{1-n}$ to obtain $\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$.

Existence and Uniqueness (2nd order) Given a function $f(x_2, x_1, t)$ suppose that f, $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ are continuous for $a_1 < x_1 < a_2, b_1 < x_2 < b_2, t_1 < t < t_2$. Then for all initial conditions $x(t_0) = x_0, \dot{x}(t_0) = y_0$ with $x_0 \in (a_1, a_2)$ and $y_0 \in (b_1, b_2)$ and $t \in (t_1, t_2)$ there exists a unique solution (a continuous function with two continuous derivatives that satisfies the initial conditions and DE) of $\ddot{x} = f(\dot{x}, x, t)$ on some interval I containing t_0.

Definition: Linear Independence The functions $x_1(t), \ldots, x_n(t)$ are linearly independent on an interval I if the only solution of $\alpha_1 x_1(t) + \cdots + \alpha_n x_n(t) = 0$, for all $t \in I$ is $\alpha_1 = \cdots = \alpha_n = 0$.

Definition: Wronskian The Wronskian of two functions x_1 and x_2 is $W[x_1, x_2](t) = \det \begin{vmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{vmatrix}$. If $W[x_1, x_2] \neq 0$ on I then the functions x_1 and x_2 are linearly independent.

Homogenous linear second order Consider $a\ddot{x} + b\dot{x} + cx = 0$. Solve $ar^2 + br + c = 0$. If the roots are distinct and real, write $x(t) = Ae^{r_1 t} + Be^{r_2 t}$. If the roots are repeated, $x(t) = Ae^{rt} + Bte^{rt}$, if the roots are complex conjugates, $r = \mu \pm i\lambda$ so $x(t) = e^{\mu t}(A\cos(\lambda t) + B\sin(\lambda t))$.

Particular equation forms

- nth order polynomial: Guess an nth order polynomial
- Exponential: If exponential is solution to homogenous equation, guess Cte^{rt}. If homogenous equation has repeated roots and exponential is the solution, guess Ct^2e^{rt}.
- Sine or cosine (or both): $x_p(t) = C\sin\omega t + D\cos\omega t$. If given trig function solves the homogenous equation, guess $x_p(t) = Ct\sin\omega t + Dt\cos\omega t$.
- Product of elementary functions: Take product of guesses.

Higher order linear equations with constant coefficients Solve homogeneous equation with guess e^{rt}, obtain roots r of auxiliary equation.

- Each non-repeated real root contributes e^{rt}.

- For roots repeated m times, we have m linearly independent solutions $e^{kt}, te^{kt}, \dots, t^{m-1}e^{kt}$.
- For non-repeated complex conjugates $\mu \pm i\lambda$, we have two solutions: $e^{\mu t}\sin\lambda t, e^{\mu t}\cos\lambda t$
- For repeated complex conjugates repeated m times, introduce $2m$ solutions: $e^{\mu t}\cos\lambda t, e^{\mu t}\sin\lambda t, te^{\mu t}\cos\lambda t, te^{\mu t}\sin\lambda t, \dots, t^{m-1}e^{\mu t}\cos\lambda t, t^{m-1}e^{\mu t}\sin\lambda t$.

Higher order linear inhomogeneous equations Given $\frac{d^n x}{dt^n}+p_1(t)\frac{d^{n-1}x}{dt^{n-1}}+\dots+p_k(t)\frac{d^{n-k}x}{dt^{n-k}}+\dots+p_{n-1}(t)\frac{dx}{dt}+p_n(t)x=0$, the general solution is the linear combination of n linearly independent functions that satisfy:

$$W[f_1,\dots,f_n](t) = \det\begin{vmatrix} f_1(t) & \cdots & f_n(t) \\ \vdots & \ddots & \vdots \\ d^{n-1}f_1/dt^{n-1}(t) & \cdots & d^{n-1}f_n/dt^{n-1}(t) \end{vmatrix} \neq 0, t\in I$$

Reduction of Order Given $a(t)\ddot{x}+b(t)\dot{x}+c(t)x=0$, suppose $u(t)$ is one known solution, then substitute $x(t)=u(t)y(t)$ and obtain $[a(t)u(t)]\ddot{y}+[2a(t)\dot{u}(t)+b(t)u(t)]\dot{y}=0$, which is a linear first order equation for $\dot{y}$.

Variation of Constants Consider $\ddot{x}+p(t)\dot{x}+q(t)x=g(t)$. Suppose we have the homogenous solution $x_c(t)=Ax_1(t)+Bx_2(t)$. Seek solutions to the inhomogeneous problem in the form $x(t)=u_1(t)x_1(t)+u_2(t)x_2(t)$. Impose condition that $\dot{u}_1(t)x_1(t)+\dot{u}_2(t)x_2(t)=0$. Then $\dot{x}=u_1\dot{x}_1+u_2\dot{x}_2$ and we end up with $\dot{u}_1\dot{x}_1+\dot{u}_2\dot{x}_2=g(t)$. Solve two equations:

$$\dot{u}_1x_1+\dot{u}_2x_2=0$$
$$\dot{u}_1\dot{x}_1+\dot{u}_2\dot{x}_2=g(t)$$

Then:

$$\dot{u}_1=\frac{-x_2 g}{W}$$
$$\dot{u}_2=\frac{x_1 g}{W}$$

where $W=x_1\dot{x}_2-x_2\dot{x}_1$. Integrate to obtain u_1 and u_2.

Cauchy-Euler Equations Given $ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0$, substitute $x=e^z$ to obtain $a\frac{d^2y}{dz^2}+(b-a)\frac{dy}{dz}+cy=0$.

Alternative solution: Try $y(x)=x^k$ to obtain the indicial equation: $ak(k-1)+bk+c=0$.

- Case 1: Two real roots. Then the general solution is $y(x)=c_1x^{k_1}+c_2x^{k_2}$.
- Case 2: Repeated real roots: Use reduction of order to find the second solution to be $x^k\ln x$. Hence $y(x)=c_1x^k+c_2x^k\ln x$.
- Case 3: Complex Roots: Write $k=\rho\pm i\omega$. Then $y(x)=x^\rho[c_1\cos(\omega\ln x)+c_2\sin(\omega\ln x)]$.

Power series solutions of second order linear equations Given $y''+p(x)y'+q(x)y=0$, write $y(x)=\sum_{n=0}^\infty a_nx^n$. Ensure solution converges by ensuring $|x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ for all $x\in I$ or calculate the radius of convergence $\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. Express $p(x)$ and $q(x)$ as power series as well $p(x)=\sum_{n=0}^\infty p_nx^n, q(x)=\sum_{n=0}^\infty q_nx^n$. If p and q are not analytic, see pages 184-186 of Robinson.

Regular Singular Equations Consider $y''+p(x)y'+q(x)y=0$. If $p(x)$ and $q(x)$ are not analytic but $xp(x)$ and $x^2q(x)$ are, then make the guess $y=x^\sigma\sum_{n=0}^\infty a_nx^n$ and find the recursion relation. Find the indicial equation: $\sigma(\sigma-1)+p_0\sigma+q_0$ where p_0 and q_0 are from writing $p(x)=\frac{p_0}{x}+p_1+p_2x+\dots$ and $q(x)=\frac{q_0}{x^2}+\frac{q_1}{x}+q_2+q_3x+\dots$.

- Case 1: σ_1,σ_2 are distinct real roots that do not differ by an integer. Then we have $c_1\sum_{n=0}^\infty a_nx^{\sigma_1+n}+c_2\sum_{n=0}^\infty b_nx^{\sigma_2+n}$.
- Case 2: $\sigma_1=\sigma_2+n, n\in\mathbb{Z}, n\geq 1$. Find the first solution $y_0(x)=\sum_{n=0}^\infty a_nx^{n+\sigma_2}$. Then the second solution will be in the form $y_1(x)=y_0(x)\ln x+\sum_{n=0}^\infty b_nx^{\sigma_2+n}$ where σ_2 is the smaller of the two roots.
- Case 3: Repeated roots. Find the first solution $y_0(x)=\sum_{n=0}^\infty a_nx^{n+\sigma}$. Then the second solution will be in the form $y_1(x)=y_0(x)\ln x+\sum_{n=1}^\infty b_nx^{\sigma+n}$. Note the sum is taken from 1.

Definitions: Vector First Order Equations For n differential equations with:

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n, t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n, t)$$

Write $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{f}(\mathbf{x}, t) = (f_1, f_2, \ldots, f_n)^T$. Then we have $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$.

The solution to this differential equation with initial value $\mathbf{x}(t_0) = \mathbf{x}_0$ on open interval containing t_0 is the continuous function $\mathbf{x} : I \to \mathbb{R}^n$ that satisfies the differential equation and initial conditions.

Uniqueness and existence for vector first order equations If $\mathbf{f}(\mathbf{x}, t)$ and $D\mathbf{f}(\mathbf{x}, t)$, where $D\mathbf{f}$ is the matrix of first partial derivatives $D\mathbf{f} = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n \partial x_n \end{pmatrix}$ are continuous functions of $\mathbf{x}$ and t with $\mathbf{x} \in U = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ and $c < t < d$, then for any $\mathbf{x}_0 \in U$ and $t_0 \in (c, d)$ there is a unique solution on some open interval containing t_0. Note: to check continuity of $D\mathbf{f}$, simply check the continuity of each of $\partial f_i/\partial x_j$ for $i, j = 1, 2, \ldots, n$.

Matrix Differential Equations Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{x}$ is a column matrix of unknown functions, if $\mathbf{A}$ has distinct real eigenvalues: Write $\mathbf{x}(t) = Ae^{\lambda_1 t}\mathbf{v}_1 + \ldots$ with eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$.

Diagonalizing Given the matrix differential equation $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, we find a diagonalising matrix of eigenvectors $\mathbf{P}$ such that we can write $\mathbf{x} = [\mathbf{v}_1\mathbf{v}_2 \cdots]\tilde{\mathbf{x}} = \mathbf{P}\tilde{\mathbf{x}}$ and $\frac{d\tilde{\mathbf{x}}}{dt} = \frac{d}{dt}\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\tilde{\mathbf{x}}$. Then the equation for $\tilde{\mathbf{x}}$ is $\frac{d\tilde{\mathbf{x}}}{dt} = diag(\lambda_1, \ldots, \lambda_n)\tilde{\mathbf{x}}$ which is uncoupled and hence we obtain $\tilde{x}_1(t) = Ae^{\lambda_1 t}$, etc. We then write the general solution for $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + \ldots$.

Phase diagrams for coupled equations Draw in the eigenvectors through the origin (in both directions). Add arrows towards the origin if the eigenvalue if negative, away from the origin if the eigenvalue is positive. Fill in the curves at the side based on whether the origin is a stable point, unstable point or saddle point.

Definition: Stable manifold The stable manifold $W^s(0)$ of the origin is all those points lying on trajectories that approach the origin as $t \to \infty$. For a stable node, it is the whole plane, for a saddle point, the eigenvector corresponding to a negative eigenvalue and for an unstable node, just the origin.

Definition: Unstable manifold The unstable manifold $W^u(0)$ of the origin is all the points lying on trajectories that would approach the origin if the sense of time were reversed. For the stable node, it is just the origin, for a saddle point, only the eigenvector corresponding to the positive eigenvalue (in both directions) and for the unstable node it is the whole plane.

Complex Eigenvalues and Eigenvectors Let $\mathbf{A}$ have eigenvectors $\boldsymbol{\eta}_- = \mathbf{v}_1 - i\mathbf{v}_2$ and $\boldsymbol{\eta}_+ = \mathbf{v}_1 + i\mathbf{v}_2$, and corresponding eigenvalues $\lambda_\pm = \rho \pm i\omega$. Then $\mathbf{A}$ can be diagonalised using $[\mathbf{v}_1\mathbf{v}_2]$ to obtain $[\mathbf{v}_1\mathbf{v}_2]^{-1}\mathbf{A}[\mathbf{v}_1\mathbf{v}_2] = \begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}$. Hence we need to solve the simpler equation $\frac{d\tilde{\mathbf{x}}}{dt} = \begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}\tilde{\mathbf{x}}$. In polar coordinates with the vertical and horizontal coordinates being $\tilde{x}$ and $\tilde{y}$ respectively, the solution to this simplified equation satisfies $\dot{r} = \rho r, \dot{\theta} = -\omega$ and hence we have $r(t) = r(0)e^{\rho t}, \theta(t) = \theta(0) - \omega t$, which spiral around the origin with angular velocity $-\omega$.

Definitions: Spirals in 2D systems If the real part of the complex eigenvalue is negative, the origin is stable and is called a **stable spiral**. If the real part of the eigenvalues is positive, then the origin is an **unstable spiral**. If the eigenvalues are purely imaginary, the orbits are circles centred at the origin and the origin is referred to as a **centre**.

Repeated Real Eigenvalue (Non-zero) We can only find one eigenvector $\mathbf{v}$ directly from the repeated eigenvalue λ. Pick any vector $\mathbf{v}_2$ not in the same direction as $\mathbf{v}$. Define $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2$, and the solution is $\mathbf{x}(t) = [Ate^{\lambda t} + Be^{\lambda t}]\mathbf{v}_1 + Ae^{\lambda t}\mathbf{v}_2$. $\mathbf{v}_1$ will be in the same direction as the eigenvector $\mathbf{v}$. Near the origin, trajectories will be parallel to the eigenvector. Far away from the origin, trajectories will also be parallel to the eigenvector, but in the other direction. Hence trajectories have to "turn around" and form an S-shaped **improper node**.

Homogeneous Difference Equations Given $ax_{n+2} + bx_{n+1} + cx_n = 0$, make the guess $x_n = k^n$ to get the auxiliary equation $ak^2 + bk + c = 0$.

- Case 1: Distinct real roots. General solution is $x_n = c_1 k_1^n + c_2 k_2^n$.
- Case 2: Repeated roots. Find the first solution $x_n = k^n$. The second solution will be nk^n. Then the general solution will be $c_1 k^n + c_2 n k^n$.
- Case 3: Complex Roots. Write $k = a \pm ib = re^{\pm i\theta}$. Then the solution is $x_n = r^n[c_1 \cos n\theta + c_2 \sin n\theta]$.

Non-homogeneous Difference Equations Given $ax_{n+2} + bx_{n+1} + cx_n = f_n$, make the following guesses for the particular solution:

- Polynomial in n. Guess a polynomial of the same order.
- Power $f_n = \lambda^n$. If λ is not a solution of the auxiliary equation, try $\alpha\lambda^n$. If λ is a non-repeated root of the auxiliary equation, try $\alpha n \lambda^n$. If λ is a repeated root, try $\alpha n^2 \lambda^n$. Solve for α.

Non-linear first order difference equations: Fixed points A fixed point is a point x^* such that $f(x^*) = x^*$.

Stable fixed points A fixed point is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - x^*| < \delta \implies |f^n(x_0) - x^*| < \epsilon$ for all $n \geq 1$.

Attracting point A point is attracting if there is a $\delta > 0$ such that $|x_0 - x^*| < \delta \implies f^n(x_0) \to x^*$ as $n \to \infty$.

Unstable fixed points A point is unstable if there exists $\epsilon > 0$ such that for all $\delta > 0$ we can find an x_0 with $|x_0 - x^*| < \delta$ but $|f^n(x_0) - x^*| > \epsilon$ for some $n > 0$.

Periodic orbits A periodic orbit of period k is a sequence of k values $\{x_1, \ldots, x_k\}$ such that $f(x_j) = x_{j+1}$ for $j = 1, 2, \ldots, n-1$ and $f(x_n) = x_1$. Also we require $f(x_j) \neq x_1$ for $j = 1, 2, \ldots, n-1$.

Coupled Non-linear Equations: Stationary points Due to uniqueness of solutions, curves in the phase diagram cannot cross each other. If two curves cross, that point must be a stationary point so that the curves do not actually pass through it.

Linearization Calculate the Jacobian matrix, then substitute the coordinates of the stationary point to find the constant coefficient matrix near each stationary point.