Chapter 1

Week 1

1.1 Lecture 31 Mar 2014

1.1.1 Ma1b Review

Inner Product Define $< x, y >= x \cdot y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$.

Norm Define $||x|| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Distance Define $d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

Triangle inequality $||x + y|| \leq ||x|| + ||y||$

Cauchy-Schwartz $||x \cdot y|| \leq ||x|| \cdot ||y||$.

1.1.2 Topology of $\mathbb{R}^n$

Convergence Suppose we have a sequence of points $x_i \in \mathbb{R}^n$. We say the sequence of points converges to a x, $x_i \to x$ iff $||x_i - x|| \to 0$ iff $\forall \epsilon > 0, \exists N \forall n \geq N, ||x_n - x|| \leq \epsilon$. Write this as $\lim_{i\to\infty} x_i = x$. Hence it follows that $|x_i| \leq ||x||$ and $||x|| \leq \sqrt{n} \max |x_i|$. Also, convergence of a sequence to a point is equivalent to pointwise convergence of the coordinates to the coordinates of the limit.

Open ball Given $a \in \mathbb{R}^n$, and $r > 0$, define $B_a(r) = \{x : ||x - a|| < r\}$ to be the open ball with center a and radius r.

Closed ball $\overline{B_a(r)} = \{x : ||x - a|| \leq r\}$

Sphere $S_a(r) = \{x : ||x - a|| = r\}$

Open Rectangular Box Given vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Define the interval between a and b to be $(a, b) = (a_1, b_1) \times \ldots \times (a_n, b_n) \in \mathbb{R}^n$. The closed box is just the same, but with closed intervals $[a, b]$.

Interior Point Let $S \subseteq \mathbb{R}^n$, $a \in S$. a is an interior point of S if there exists $r > 0$ such that $B_a(r) \subseteq S$. Equivalently, a is an interior point of S iff there is an open box B so that $a \in B$ and $B \subseteq S$. The set of all interior points of a set is the interior of the set. Call this $Int(S)$.

Open Set A set $S \subseteq \mathbb{R}^n$ is open if every point of S is an interior point. Equivalently, S is the union of open balls/boxes. By definition, the empty set is open. Also, the whole space $\mathbb{R}^n$ is open. Note also that $(a, b]$ is not open. However, an open set $U \in \mathbb{R}^n$ is not open when viewed as a subset of a higher dimension space $\mathbb{R}^{n+1}$.

Exterior point A point $a \in \mathbb{R}^n$ is in the exterior of S if $a \in Int(\mathbb{R}^n \backslash S)$, if it is in the complement of S. Denote the exterior of set S as $Ext(S)$.

Boundary Point A point $a \in \mathbb{R}^n$ is at the boundary of $S \subseteq \mathbb{R}^n$ if $\forall r > 0$, $B_a(r)$ has points in S and outside S. Denote the set of all these points as ∂S. Note that $\partial S = \mathbb{R}^n \backslash (Int(S) \cup Ext(S))$.

Closed set A subset S is closed if it contains all its boundary points. Note that if $S \subseteq \mathbb{R}^n$ is closed iff $S \subseteq \mathbb{R}^{n+1}$ is closed in $\mathbb{R}^{n+1}$. The following are equivalent (TFAE): (i) S is closed (ii) $\mathbb{R}^n \backslash S$ is open (iii) S is closed under limits: for $a_n \in S, a_n \to a \implies a \in S$.

Proof Assume S is closed. If $a \notin S$, then there is a positive real number $r > 0$ such that $B_a(r) \cap S = \phi$, so $\mathbb{R}^n \backslash S$ is open. Hence $(i) \implies (ii)$. Now let $a \in \partial(S)$, then $a \notin \mathbb{R}^n \backslash S$ since the complement is open, so $a \in S$. Hence $(ii) \implies (i)$. If $a_n \in S$ and $a_n \to a$, then $a \notin Ext(S)$. Hence $(i) \implies (iii)$. If $a \in \partial S$, then for each n, there exists $a_n \in B_a(1/n) \cap S$. Hence $(iii) \implies (i)$.

Basic Facts about Open and Closed Set (i) If we have a collection of open sets, then their union is also open. (ii) If there are finitely many open sets, then their intersection is also open. Note that if there are infinitely many sets, (ii) does not apply. Consider the intervals $(-1/n, 1/n)$. THen the intersection of infinitely many such sets over the natural numbers is just the zero element, and this is not open.

1.2 Lecture 02 Apr 2014

De Morgan's Law $\mathbb{R}^n \backslash \cap_i S_i = \cup_i (\mathbb{R}^n - S_i)$ and $\mathbb{R}^n \backslash \cup_i S_i = \cap_i (\mathbb{R}^n - S_i)$.

All finite sets are closed

Neither open nor closed The interval $(0, 1]$ is neither open nor closed.

Closure The closure of a set S, $\overline{S}$, is the union of the set and its boundary. $\bar{S} = S \cup \partial(S)$. Hence the closure of $(0, 1]$ includes zero, hence it is $\overline{(0, 1]} = [0, 1]$. The closure of an open ball is simply the closed ball. Can think of the closure as the set including all the limits of convergence of the set. If S is a closed set, it equals its own closure.

Theorem from Ma1a Let $S \subseteq \mathbb{R}$ be closed and bounded. Then S has both a maximum point and a minimum point.

Proof Since S is bounded, let a be the greatest lower bound (inf) of S and let b be the least upper bound of S (sup). We will show that $a, b \in S$, so S contains the minimum and maximum. Consider a. Then for every $\epsilon > 0$, $S \cap [a, a + \epsilon) \neq \phi$ there has to be some elements ϵ away from a, since a is the limit. Also, $(a - \epsilon, a] \cap (\mathbb{R} - S) \neq \phi$, that is, there are also some elements outside of S when you decrease a slightly.

Compact A set $C \subseteq \mathbb{R}^n$ is compact if for every arbitrary covering of C by open sets, $C \subseteq U_1 \cup U_2, \ldots$, C is contained within finitely many of them $C \subseteq U_1 \cup \ldots \cup U_n$. Example: finite sets are compact. This is because every member of the finite set is part of some set. Let m_i be such that $a_i \in U_{m_i}$. Let $m = \max(m_1, \ldots, m_k)$. Then $\{a_1, \ldots, a_k\} \subseteq U_1 \cup \ldots \cup U_m$. Another example: The closed ball is compact, but the open interval $(0, 1)$ is not compact. This is because I can see $(0, 1)$ as the infinite union of intervals $(1/m, 1 - 1/m)$ for $m > 0$. However, any finite subcovering will not contain $(0, 1)$, since you can always find elements in $(0, 1)$ that are outside the finite collection of sets.

Bounded A set $S \subseteq \mathbb{R}^n$ is bounded for some large r if $S \subseteq B_0(r)$ for some open ball centered at the origin. Otherwise, it is unbounded.

Heine-Borel Theorem A set $C \subseteq \mathbb{R}^n$ is compact iff C is closed and bounded. Hence the closed ball/box is compact.

Proof $\implies$: Assume C is compact. Then C is contained within the infinite union of open balls $B_0(1) \cup B_0(2) \ldots = \mathbb{R}^n$. By compactness, C is contained within finitely many of them, that is $C \subseteq B_0(m)$, for some m, since all previous open balls are contained within $B_0(m)$. Hence C is bounded. To show that C is closed, we show that its complement $\mathbb{R}^n - C$ is open. Pick some point a outside C. Let $V_m = \mathbb{R}^n \backslash \overline{B_a(1/m)}$, which is open. Also, $V_1 \subseteq V_2 \ldots$ and $C \subseteq \cup_m V_m$, since the intersection of all the balls is the single point a. By compactness, C is containing in finitely many of those sets, hence it is contained in V_m for some m (V_m contains all the sets before it). So $C \cap B_a(1/m) = \phi$ or $B_a(1/m) \subseteq \mathbb{R}^n \backslash C$. Hence we have found an open ball centered at a that is disjoint with C. Hence the complement of C is open, and hence C is closed.

$\leftarrow$: It will suffice to prove that every closed box $B = [a, b] = [a_1, b_1] \times \ldots \times [a_n, b_n]$ is compact. This is because if C is closed and bounded, then $C \subseteq B$ for some closed box B. Let $C \subseteq U_1 \cup U_2 \ldots$ with each U_i open be a covering of C. Then $B \subseteq (\mathbb{R}^n - C) \cup U_1 \cup U_2 \ldots$, since any point in B is either in C or not in C. If it is in C, it will be contained within the covering.

If it is not in C then it will be contained in $\mathbb{R}^n - C$. But $\mathbb{R}^n - C$ is open. Hence for some m, $B \subseteq (\mathbb{R}^n - C) \cup U_1 \cup \ldots \cup U_m$ so $C \subseteq U_1 \cup \ldots U_m$.

Any closed interval is compact It will suffice to show that the unit interval in the real line is compact. Split this interval into two subintervals of equal length. Call these I_0 and I_1. Repeat this process to obtain $I_{00}, I_{01}, I_{10}, I_{11}$. Then we have closed intervals indexed by binary numbers. Assume that I is contained within an infinite union of open sets U_i. Assume, to the contrary, that the interval cannot be contained within finitely many of the open sets. Say a binary sequence S is bad if we cannot contain I_s by finitely many U_is. Hence if an interval is bad, and we split it into half, at least one of the two halves is going to be bad. If S is bad, then one of S0 or S1 is also bad. So we can find an infinite sequence of zeroes and ones such that the binary number represented by that sequence corresponds to a bad interval. Then taking the intersection of all these intervals with length going to zero, it will converge to a single point. Call this point x. Then x belongs to some U_i, since x was in the original interval, which was covered by the infinite union of U_i. Since U_i is open, there exists some interval around x that is completely contained in U_i. But that means that the interval obtained by repeated halving can be contained within U_i, which is an open set. But this means that the interval cannot be bad. Contradiction. This means that the original interval cannot be bad. To do this in higher dimensions, we do this similarly. For example, in $\mathbb{R}^2$, we cut the interval into 4 pieces instead. Hence in general, in $\mathbb{R}^n$, we cut the interval into 2^n subintervals.

1.3 Recitation 03 Apr 2014

OH: Sloan 357, Sunday 9pm

Topology of $\mathbb{R}^n$ 1. Convergence: $\forall \epsilon > 0, \exists N(\epsilon) s.t. \forall n > N(\epsilon), ||x_n - x|| < \epsilon$.

1.4 Lecture 04 Apr 2014

Notation Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, with the domain D and range $f(D) = \{f(x) : x \in D\}$. If $m \neq 1$, then we can write the function in terms of its components $f(x) = (f_1(x), \ldots, f_m(x))$, and define the scalar field $f_i(x)$ be the ith coordinate of $f(x)$ with $f_i : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

Limits Let $a \in \mathbb{R}^n$ be such that $\forall r > 0, (B_a(r) \backslash \{a\}) \cap D \neq \phi$ so that there is some point in the domain arbitrarily close to a. Note that we do not assume that a is part of the domain D. Let $b \in \mathbb{R}^m$ such that $\lim_{x \to a} f(x) = b$. This means that $\forall \epsilon > 0, \exists \delta > 0$ such that whenever $x \in D$ and $0 < ||x - a|| < \delta$, then $||f(x) - b|| < \epsilon$. In terms of components $\lim_{x \to a} f(x) = b$ iff $\forall 1 \leq i \leq m$, $\lim_{x \to a} f_i(x) = b_i$, when $b = (b_1, \ldots, b_m)$.

Continuity Let $a \in D$. Then $f(a)$ is defined. We say that f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. Note that a cannot be an isolated point of D. This means that there are points in the domain that are arbitrarily close to a. An isolated point a is such that there exists an n-ball around it such that the points in the n-ball are not in the domain. But by convention, we consider that f is continuous at any isolated point of the domain. Equivalently, f is continuous at $a \in D$ iff for any sequence of points $x_n \in D$ such that $x_n \to a$, we have that $f(x_n)$ converges to $f(a)$. f is continuous at a iff each f_i is continuous at a also. f is continuous at $A \subseteq D$ iff f is continuous at every point $a \in A$. In particular, f is continuous iff it is continuous on D.

Continuity preserved under composition Let $g : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f : E \subseteq \mathbb{R}^m \to \mathbb{R}^k$ and $a \in D, g(a) \in E$. Then if g is continuous at a, and f is continuous at $g(a)$, then the composition of the functions $f \circ g(x) = f(g(x))$ is continuous at a. Prove by considering a sequence $x_n \in D$ that approaches a. Then consider the sequence $g(x_n)$ and $f(g(x_n))$.

Example: Projection Function Consider $p_i : \mathbb{R}^n \to \mathbb{R}$, such that $p_i(x_1, \ldots, x_n) = x_i$. p_i is continuous.

Example: Linear functions All linear functions are continuous. Note that $T(x) - T(a) = T(x - a)$ so it suffices to check continuity at 0. Taken an element $L = (L_1, \ldots, L_n) \in \mathbb{R}^n$ or $L = L_1 e_1 + \ldots + L_n e_n$ where e_i is the standard basis of $\mathbb{R}^n$. We apply the linear transformation $T(L) = L_1 T(e_1) + \ldots + L_n T(e_n)$. If $L \to 0$, then $L_i \to 0$, so $T(L) \to 0$. Hence T is continuous at zero, and by linearity, it is continuous at every point.

Example: Polynomials Addition and multiplication defined on $\mathbb{R}^2 \to \mathbb{R}$ are continuous, so by composition and the continuity of projection functions, every polynomial $P(x_1, \ldots, x_n) = \sum c_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n}$ is continuous.

Example: Rational functions Rational functions $\frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)}$ where P and Q are continuous are continuous whenever the denominator are not zero.

Equivalent formulation of continuity Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous (at every point) iff for every open $U \subseteq \mathbb{R}^m$, the preimage $f^{-1}(U) = \{x \in \mathbb{R}^n, f(x) \in U\}$ is open relative to D. Equivalently, if U is closed, the preimage must be closed relative to D for f to be continuous.

Proof ($\implies$) If $a \in f^{-1}(U)$, then $f(a) \in U$, so $\exists \epsilon > 0$ such that $B_{f(a)}(\epsilon) \in U$. By continuity, $\exists \delta > 0$ such that $f(B_a(\delta)) \subseteq B_{f(a)}(\epsilon)$, so $B_a(\delta) \subseteq f^{-1}(U)$. ($\leftarrow$:) Let $a \in \mathbb{R}^n$. Fix $\epsilon > 0$ in roder to find $\delta > 0$ such that $f(B_a(\delta)) \subseteq B_{f(a)}(\epsilon)$. Now $B_{f(a)}(\epsilon)$ is open in $\mathbb{R}^m$, so by assumption the preimage of that set $f^{-1}(B_{f(a)}(\epsilon))$ is open in $\mathbb{R}^n$ and contains a, so $\exists \delta > 0$ such that $B_a(\delta) \subseteq f^{-1}(B_{f(a)}(\epsilon)$ so $f(B_a(\delta)) \subseteq B_{f(a)}(\epsilon)$.

Level Set and continuity $L_c(f) = \{x \in \mathbb{R}^n : f(x) = c\} = f^{-1}(\{c\})$. If f is continuous, then $L_c(f)$ is closed.

Open Relative to Set Let $D \subseteq \mathbb{R}^n$. We say that $A \subseteq D$ is open relative to D if for each element $a \in A$ there is $r > 0$ such that $B_a(r) \cap D \subseteq A$. Equivalently, this means that A is open in D iff $A = D \cap U$ where U is an open set in $\mathbb{R}^n$.

Closed relative to set Similarly, define for $A \subseteq D$ to be closed in D iff A is the intersection of D with a closed set in $\mathbb{R}^n$.

Continuity and Compactness Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be continuous and C compact. Then the image $f(C)$ is compact.

Proof Let $f(C) \subseteq U_1 \cup \ldots$ be a subset of an infinite collection of open sets. Let $f^{-1}(U_i) = C \cap W_i$ with W_i open in $\mathbb{R}^n$. This follows because f is continuous. Then $C \subseteq W_1 \cup \ldots$, so there is k such that $C \subseteq W_1 \cup \ldots \cup W_k$ so $C \subseteq f^{-1}(U_1) \cup \ldots \cup f^{-1}(U_k)$ so $f(C) \subseteq U_1 \cup \ldots \cup U_k$. So $f(C)$ is covered by a finite subcovering.

Theorem If $f : C \subseteq \mathbb{R}^n \to \mathbb{R}^m$, with f continuous and C compact, then f has a minimum and maximum point in C, such that $\exists x_{min}, x_{max} \in C$ such that $\forall x \in C, f(x_{min}) \leq f(x) \leq f(x_{max})$. Note that $f(C)$ is a compact subset of the reals, since f is a scalar field. Hence we can immediately use the sup and inf of a set of real values.

Chapter 2

Week 2

2.1 Lecture 7 April 2014

Review of 1 variable differentiation Let $f : D \subseteq \mathbb{R} \to \mathbb{R}, a \in Int(D)$ Then define $\lambda = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ when the limit exists. Rewrite this as $\lim_{h\to 0} \frac{f(a+h)-f(a)-\lambda h}{h} = 0$. This is the same as when the absolute values go to zero.

Multivariable differentiation Now let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then let λ be a linear function such that λh is a good approximation of $f(a+h) - f(a)$.

Differentiability A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $a \in Int(D)$ with derivative $L : \mathbb{R}^n \to \mathbb{R}^m$ which is a linear transformation, if $\lim_{h\to 0} \frac{||f(a+h)-f(a)-L(h)||}{||h||} = 0$. This is defined because if h is small enough, it will be within the open ball at a. If the derivative of f at a exists, we call it the total derivative, and denote it by $f'(a)$ or $T_a f$.

Uniqueness of total derivative Such an L, if it exists, is unique.

Properties of the total derivative $f(a+h) = f(a) + f'(a)(h) + ||h||E(a,h)$, where $E(a,h)$ can be thought of as the error of the approximation. $E(a,h) \in \mathbb{R}^m$.

Directional Derivative Define the directional derivative for $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m, a \in Int(D), u \neq 0, ||u|| = 1, u \in \mathbb{R}^n$ to be $f'(a,u) = \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t} \in \mathbb{R}^m$ if it exists. For a vector-valued function, $f'(a,u)$ exists iff $f_i'(a,u)$ exists for all $i = 1, \dots, m$.

Partial Derivative The ith partial derivative of f, $f'(a, e_i), \frac{\partial f}{\partial x_i}$ or $D_i f(a)$, is the directional derivative with u being the ith unit vector. For a vector valued function, we can write this as $D_i f(a) = (D_i f_1(a), \dots, D_i f_m(a))$.

Proposition If f is differentaible at a, then all its directional derivatives exist, and the directional derivative $f'(a,u)$ is obtained from the total derivative operating on u: $f'(a,u) = f'(a)(u)$.

Proof Let L satisfy the definition of the total derivative. Then for every $t \in \mathbb{R}$, if $h = tu$, then $\frac{||f(a+tu)-f(a)-L(tu)||}{|t|||u||} \to 0$ as $t \to 0$. Since $L(tu) = tL(u)$ since L is a linear transformation, for $t > 0$, we can multiply by $||u|| \neq 0$ to get $\frac{f(a+tu)-f(a)}{t} - L(u) \to 0$ as $t \to 0$. For $t < 0$, we multiply by $-||u|| \neq 0$ so that $\frac{f(a+tu)-f(a)}{t} - L(u) \to 0$ as $t \to 0$. Hence $f'(a,u) = L(u)$.

Computing the total derivative Since the total derivative is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, it will be an $m \times n$ matrix in matrix form. Choose the standard basis $(e_1, \dots, e_n)$ for $\mathbb{R}^n$ and $(d_1, \dots, d_m)$ for $\mathbb{R}^m$. We simply evaluate the derivative at the standard basis vectors. Let $v = \sum_{j=1}^n \alpha_j e_j \in \mathbb{R}^n$. Then $f'(a)(v) = f'(a)\sum_{j=1}^m \alpha_j e_j = \sum_{j=1}^m \alpha_j f'(a)(e_j) = \sum_{j=1}^m \alpha_j \frac{\partial f}{\partial x_j}(a) = \sum_{j=1}^m \alpha_j \sum_{i=1}^n \frac{\partial f_i}{\partial x_j}(a) d_i$. We can reverse the order of summation to get $f'(a)(v) = \sum_{i=1}^n \sum_{j=1}^m \alpha_j \frac{\partial f_i}{\partial x_j}(a) d_i$. Hence $(f'(a))_{ij} = \frac{\partial f_i}{\partial x_j}(a)$. Call this the Jacobian matrix, $Df(a)$.

2.2 Lecture 09 April 2014

Special cases for the total derivative: Scalar Field Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. Then the Jacobian matrix is a $1 \times n$ matrix containing the list of partial derivatives of f. This is a row vector in $\mathbb{R}^n$, hence it can be viewed as an element in $\mathbb{R}^n$. This is called the gradient of f. Note that we can write the operation of the linear operator on an arbitrary vector v as an inner product: $f'(a,u) = f'(a)(u) = \nabla f(a) \cdot u$. Note further that $\nabla f \cdot u = ||\nabla f|| \cos\theta$ if u is a unit vector. Hence ∇f points in the direction of maximum rate of increase increase of f, and this maximum value is $||\nabla f||$.

Special case: n=1 Consider $f : D \subseteq \mathbb{R} \to \mathbb{R}^m$. Hence we can view $f(t)$ as the position of a moving particle at time $t \in \mathbb{R}$. Then $f'(t)$ is an $m \times 1$ column vector with the entries as the derivatives of each component of f. This can be viewed as a vector in $\mathbb{R}^m$, which is the velocity vector. When this is non-zero, this vector is the tangent line to the parametrized curve $f(t)$ at t.

Proposition Let $f \in D \subseteq \mathbb{R}^n \to \mathbb{R}^m, a \in Int(D)$. If f is dfferentiable at a, then f is continuous at a.

Proof Let L be the derivative of f. Then for a $h \neq 0$, then $f(a+h) - f(a) = \frac{||h||(f(a+h)-f(a)-L(h)}{||h||} + L(h)$. But by the definition of the derivative, the $\frac{f(a+h)-f(a)-L(h)}{||h||} \to 0$ as $h \to 0$. Also, $L(h) \to 0$ as $h \to 0$ also, since L is a linear transformation. Hence $\lim_{h\to 0} f(a+h) - f(a) = 0$, and the function is continuous at a.

Theorem Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $a \in Int(D)$, and let $\epsilon > 0$ be such that $B_a(\epsilon) \subseteq D$. Let all the partial derivatives of f exist in $B_a(\epsilon)$ and are continuous at a. Then $f'(a)$ exists.

Definition: Continuously Differentiable Let $f : D \subseteq \mathbb{R} \to \mathbb{R}^m$, with D open. f is continuously differentiable or C^1 if all the partial derivatives exist in D and are continuous in D. Hence the total derivative exists at each point in D. Note that even in 1-dimension, even if f is differentiable, f does not need to be C^1, since the derivatives can be discontinuous.

Higher order partial derivatives Consider $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, with D open, and assume all partial derivatives of f exist in D. Then we can form the partial derivatives of the partial derivatives.

2.3 Recitation 10 Apr 2014

Theorem Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in Int(D)$. Suppose all the partial derivatives of f exists at a and are continuous at a. Then the total derivative of f at a exists.

Proof Let $a = (a_1, \dots, a_n)$. Also consider a vector $u = (u_1, \dots, u_n)$. Then consider the linear map $L_i(u) = \sum_{j=1}^n u_i \frac{\partial f}{\partial x_j}(a)$ (which is actually the ith row vector of u multiplied by the Jacobian of f at a). Now write $f(a+u) - f(a) = \sum \phi_j(a_j + u_j) - \phi_j(a_j)$, where $\phi_j(t) = f(a_1 + u_1, a_{j-1} + u_{j-1}, t, a_{j+1}, \dots, a_n)$. So $\phi_j(t)$ includes only the first few components of u. So in fact we are building up $f(a+u)$ from $f(a)$ in terms of component by component in u. Each element in the sum is basically moving in one particular standard unit vector direction. Hence we can apply the mean value theorem on each element in the sum. In effect, we are saying:

$$\begin{aligned}
f(a+u) - f(a) &= f(a_1 + u_1, a_2 + u_2, \dots, a_n + u_n) - f(a_1, a_2, \dots, a_n) \\
&= f(a_1 + u_1, a_2, \dots, a_n) - f(a_1, a_2, \dots, a_n) \\
&+ f(a_1 + u_1, a_2 + u_2, \dots, a_n) - f(a_1 + u_1, a_2, \dots, a_n) \\
&+ \cdots \\
&+ f(a_1 + u_1, a_2 + u_2, \dots, a_{n-1} + u_{n-1}, a_n + u_n) - f(a_1 + u_1, a_2 + u_2, \dots, a_{n-1} + u_{n-1}, a_n)
\end{aligned}$$

Argh just refer to Apostol for the proof.

2.4 Lecture 11 Apr 2014

Clairaut's Theorem Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $c \in Int(D)$, and $\epsilon > 0$ such that $B_c(\epsilon) \subseteq D$. Let $1 \leq i, j \leq n$. If all the partial derivatives $D_i f, D_j f$ and the mixed derivatives $D_{ij} f, D_{ji} f$ exist in $B_c(\epsilon)$ and the mixed derivatives are continuous at c, then the mixed derivatives are the same at the point c.

Proof We can assume $n = 2$, where we fix the values of all other variables, and only allow x_i and x_j to vary. We can also assume that $m = 1$, since we can treat f in terms of its coordinates, which are scalar functions. Hence we consider the function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $c = (a,b) \in D$, and $\epsilon > 0$ such that $B_c(\epsilon) \subseteq D$. Now take $h, k > 0$ sufficiently small such that we can form a rectangle in 2-space with coordinates $(a,b), (a+h,b), (a,b+k), (a+h,b+k)$ such that all the points in the rectangle are contained within $B_c(\epsilon)$. Define the difference $\Delta(h,k) = f(a+h,b+k) - f(a+h,b) + f(a,b) - f(a,b+k)$.

Rewriting this, $\Delta(h,k) = [f(a+h,b+k) - f(a+h,b)] - [f(a,b+k) - f(a,b)]$. Choose a point $x \in [a, a+h]$. Consider the difference $G(x) = f(x,b+k) - f(x,b)$ as a function of one variable. Then $\Delta(h,k) = G(a+h) - G(a)$. We examine $G'(x) = \frac{\partial f}{\partial x}(x,b+k) - \frac{\partial f}{\partial x}(a,b)$. Now we can apply the mean value theorem to obtain that there exists some point $x_1 \in (a, a+h)$ such that $G(a+h) - G(a) = G'(x_1)h$. Hence $\Delta(h,k) = h[\frac{\partial f}{\partial x}(x_1, b+k) - \frac{\partial f}{\partial x}(x_1, b)]$. Now consider the function $G_1(y) = \frac{\partial f}{\partial x}(x_1, y)$. Hence we can write $\Delta(h,k) = h[G_1(b+k) - G_1(b)]$. Using the mean value theorem one more time, $\Delta(h,k) = h[kG_1'(y_1)] = hk\frac{\partial^2 y}{\partial y \partial x}(x_1, y_1)$ for some $y_1 \in (b, b+k)$. Hence, $\Delta(h,k) = hk\frac{\partial^2 f}{\partial y \partial x}(x_1, y_1)$ for some $(x_1, y_1) \in (a, a+h) \times (b, b+k)$. Now we notice that we can exchange the roles of x and y to obtain that $\Delta(h,k) = hk\frac{\partial^2 f}{\partial x \partial y}(x_2, y_2)$ for some $(x_2, y_2) \in (a, a+h) \times (b, b+k)$. Notice this provides the two-dimensional mean value theorem. Hence, we can compare the equations for $\Delta(h,k)$, cancel out hk since it is a positive non-zero number, to obtain that $\frac{\partial^2 f}{\partial x \partial y}(x_2, y_2) = \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1)$. Now we can take the limit as $(h,k) \to 0$. Then $(x_1, y_1) \to (a,b) = c$ and $(x_2, y_2) \to (a,b) = c$ also. Since the mixed partial derivatives are continuous at c, we have that the mixed partial derviatives are also equal at c.

Review: Mean value theorem For F continuous in $[a, a+l], l \geq 0$, and F' exists in $(a, a+l)$, then $F(a+l) - F(a) = F'(y) \cdot l$ for some $y \in (a, a+l)$.

Notation If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with D open, we say that f is in C^2 if all the first and second order partial derivatives exist in D and are continuous in D. If a function is C^2, then the mixed derivatives are equal. Similarly, we can define partial derivatives of order larger than 2, and we define a function to be in $C^k, k \geq 1$ if all the 1st to kth order partial derivatives exist in D and are continuous in D. Then after repeated application of Clairauts Theorem, we have that if f is in C^k, then all the mixed partial derivatives of order less than order equal to k are equal provided they contain the same variables with the same multiplicity (since a single variable may occur more than once). Hence the order of differentiation does not matter for these functions (up to kth order derivatives). If f has continuous derivatives of any order, then we say that f is in C^∞ or smooth.

Chapter 3

Week 3

3.1 Lecture 14 Apr 2014

1D Chain Rule Given a function $h(x) = f(g(x)), h = f \circ g$. Then $h'(a) = f'(g(a))g'(a)$ in one-dimension.

Multivariable chain rule Consider the function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $g : E \subseteq \mathbb{R}^k \to \mathbb{R}^n$. Assume that $a \in Int(E)$ and $g(a) \in Int(D)$, and assume that $g'(a)$ and $f'g(a))$ exist. Then the derivative of the composition $h = f \circ g$ exists and is equal to $h'(a) = f'(g(a)) \circ g'(a)$. In terms of Jacobian matrices, $Dh(a) = Df(g(a)) \cdot Dg(a)$.

Special Case 1 $f : D\mathbb{R}^n \to \mathbb{R}$ is a scalar field. Write $f(y_1, \ldots, y_n)$. Also assume $g : E \subseteq \mathbb{R}^k \to \mathbb{R}^n$. Write $g(x_1, \ldots, x_k)$, which are actually n scalar functions. Then g corresponds to a change of variables $y_i = g_i(x_1, \ldots, x_k)$. Given $h = f \circ g$, $h(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_n))$ and $h : E \subseteq \mathbb{R}^k \to \mathbb{R}$. Then, after multiplying the Jacobians, $\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_i} + \ldots + \frac{\partial f}{\partial y_n}\frac{\partial g_n}{\partial x_i}$. Note that $\frac{\partial f}{\partial y_n}$ is evaluated at $g(a)$ and $\frac{\partial g_n}{\partial x_i}$ is evaluated at a.

Polar Coordinates Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, which we can write as $f(x, y)$. Now consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(r, \theta) = (r\cos\theta, r\sin\theta)$. Consider the composition $h = f \circ g$, $h(r, \theta) = f(r\cos\theta, r\sin\theta)$. Then $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$ and similarly for θ.

Special Case 2 Consider $f : D\mathbb{R}^n \to \mathbb{R}$, $g : E \subseteq \mathbb{R} \to \mathbb{R}^n$ Then $h = f \circ g : E \subseteq \mathbb{R} \to \mathbb{R}$. Then $h(t) = f(g_1(t), \ldots, g_n(t))$. Applying the chain rule, $h'(t) = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial t} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial t}$. But each g_k is a single valued function. Hence we can write this as the derivative. $\frac{dh}{dt} = (\nabla f) \cdot g'(t)$, with ∇f evaluated at $g(t)$.

Definition: Path and Curve A path, or parametrized curve is a continuous map $\alpha : I \to \mathbb{R}^n$ from an interval I in $\mathbb{R}$. Note that I can be closed or open. The range of the curve $\alpha(I)$ is the set of all points in $\mathbb{R}^n$ associated with the interval is the curve. For example, consider the unit circle. Then $I = [0, 2\pi]$, and the path is $\alpha(t) = (\cos t, \sin t)$. The curve is the unit circle. Note that the same curve can be parametrized by different paths. For instance $\beta(T) = (\cos 2t, \sin 2t)$ is a different path that traces out the same curve.

Examples The cycloid is defined by the path $\alpha(t) = (t - \sin t, 1 - \cos t), t \geq 0$. It is the path that a point on a unit circle traces as the circle rolls.

Examples The helix in $\mathbb{R}^3$ is defined by the path $\alpha(t) = (\cos t, \sin t, t)$.

Compactness Note that if the interval I is closed and bounded, then it is compact. Hence the image of I, $\alpha(I)$, is also compact. However, if I is an open interval, then the image can be unbounded.

Velocity If $\alpha'(t_0)$ exists at some point t_0, we call it the velocity or tangent vector of α at t_0. If $\alpha'(t_0) \neq 0$, then the 1-dimensional space $\lambda\alpha'(t_0), \lambda \in \mathbb{R}$ is the tangent space of α at t_0. The line $\alpha(t_0) + \lambda\alpha'(t_0)$ is the tangent line at t_0. Also, for small h, $\alpha(t_0 + h) \approx \alpha(t_0) + h\alpha'(t_0)$.

Tangent Space to Level Sets Take a scalar field $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}$. Then the level set of f at c is $L_c(f) = \{x \in D : f(x) = c\}$. Note that if f is continuous, then the level set $L_c(f)$ is a closed set of $\mathbb{R}^n$.

3.2 Lecture 16 Apr 2014

Proposition Given $f : D \in \mathbb{R}^m \to \mathbb{R}$, with D open. Consider a point $c \in \mathbb{R}$. Let $a \in L_c(f)$ such that $f(a) = c$. Now consider a path $\alpha : I \to \mathbb{R}^n$ which is differentiable and let $t_0 \in Int(I)$ be such that $\alpha(t_0) = a$ and $\alpha(I) \subseteq L_c(f)$. Then if the gradient of the function $\nabla f(a)$ exists, then it is orthogonal to the tangent of the path at this point: $\alpha'(t_0)$. That is, $\nabla f(a) \cdot \alpha'(t_0) = 0$. Note that this is applicable for any path that passes through a, and is completely contained in $L_c(f)$.

Proof Take any point $t \in Int(I)$. Then $f(\alpha(t)) = c$, since $\alpha(t)$ is on the level curve. Note c is a constant. Now we differentiate both sides with respect to t. Then we obtain $\frac{dc}{dt} = 0 = \nabla f(a) \cdot \alpha'(t_0)$ by the chain rule.

Definition: Affine tangent space If $\alpha \in L_c(f)$ and $\nabla f(a) = f'(a) \neq 0$, we call $\{x \in \mathbb{R}^n : \nabla f(a) \cdot x = 0\}$ the tangent space of $L_c(f)$ at a point a. We call the affine space $\{x \in \mathbb{R}^n : \nabla f(a) \cdot (x - a) = 0\}$ the affine tangent space attached to a. This is the translated subspace. Note that if we take any differentiable path passing through a, then the tangent at a lies in the tangent space. Equivalently, for every vector in the tangent space, there is a path such that its tangent at a is the vector in the tangent space.

Example: 3D Suppose we have a function $f : \mathbb{R}^3 \to \mathbb{R}$, then the tangent plane attached to $a \in L_c(f)$ is defined by $\nabla f(a) \cdot (x - a) = 0$, assuming $Df(a) \neq 0$. Writing this out in components, $\frac{\partial f}{\partial x}(a)(x - a_1) + \frac{\partial f}{\partial y}(a)(y - a_2) + \frac{\partial f}{\partial z}(a)(z - a_3) = 0$.

Remark Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$. The graph of this function is a surface in 3 dimensions: $\{(x, y, z) : z = g(x, y)\}$. But this a level set in $\mathbb{R}^3 : L_0(f)$, where $f(x, y, z) = z - g(x, y)$.

Taylor's Theorem for Scalar Fields Recall Taylor's Theorem in 1 variable: $f(a + h) = f(a) + \sum_{k=1}^{m-1} \frac{f^{(k)}(a)h^k}{k!} + \frac{f^{(m)}(u)}{m!}h^m$. f is defined in some open interval contained in $[a, a + h]$, and has continuous derivatives up to order m, and $u \in [a, a + h]$. Now consider the scalar field version: Consider the function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open, $a \in D$ where a is a point at which all the derivatives below exist. Define for any $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, with $f'(a, h) = f'(a)(h) = \nabla f(a) \cdot h = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)h_i$. The second order is $f''(a, h) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)h_i h_j$. Similarly, we define the third derivative to be $f'''(a, h) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(a)h_i h_j h_k$. Now by the definition of the total derivative, we have that $f(a + h) = f(a) + f'(a, h) + ||h|| \cdot E_1(a, h)$, where $E_1(a, h)$ is an error that goes to zero when $||h|| \to 0$. Taylor's theorem gives us the higher order approximations.

Taylor's Theorem Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open and $a \in D$, with $h \in \mathbb{R}^n$ such that the line connecting a and a+h is contained in D: $[a, a + h] \subseteq D$. Also assume that f is in C^n. Then there exists $u \in [a, a + h]$ such that $f(a + h) = f(a) \sum_{k=1}^{m-1} \frac{f^{(k)}(a,h)}{k!} + \frac{f^{(m)}(u,h)}{m!}$.

Proof (Sketch) Define the function of one variable $g(t)$, which is $g(t) = f(a + th)$. Then this function is defined on some open interval that contains the interval $[0, 1]$. By Taylor's Theorem for the one-variable function f, we have that $g(1) = f(a + h) = g(0) + \sum_{k=1}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(\theta)}{m!}$, where $\theta \in [0, 1]$. Note that we omit the 1^k in the Taylor expansion. We examine the derivatives of g: $g'(t) = f'(a + th)(h) = \nabla f(a + h) \cdot h$ by the chain rule. Writing this explicitly, we have that $g'(t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a + th)h_i$. Evaluating this at $t = 0$, we have that $g'(0) = f'(a, h)$. For the second derivative, we apply the chain rule to each $\frac{\partial f}{\partial x_j}$ instead of f. Then we obtain $g''(t) = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a + th)h_i h_j$. Then evaluate at zero to obtain $f''(a, h)$.

Corollary $f(a + h) = f(a) + \sum_{k=1}^m \frac{f^{(k)}(a,h)}{k!} + ||h||^m \cdot E_m(a, h)$, where $E_m(a, h) \to 0$ as $h \to 0$ is the error associated with the approximation. We obtain $E_m(a, h)$ from the Taylor Formula by adding and subtracting $\frac{f^{(m)}(a,h)}{m!}$. We note that $E_m(a, h)$ goes to zero, since $f^{(m)}$ is continuous at a.

3.3 Lecture 18 Apr 2014

Extrema of functions A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. We say f has a local maximum at $a \in D$ if $\exists r > 0$ such that $x \in B_a(r) \cap D \implies f(x) \leq f(a)$. Define the local minimum similarly. f has an absolute/global maximum at some $a \in D$ if $f(x) \leq f(a), \forall x \in D$. Define the notion of the absolute/global minimum similarly. Suppose the function is continuous and defined on some compact set. Then there will be an absolute minimum and an absolute maximum.

Theorem Consider the function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open, and let $a \in D$ be a local extremum (either maximum or minimum). Then if the function is differentiable (i.e. total derivative exists) at this point, it has to be zero. I.e. the gradient is zero, so all the partial derviatives at this point vanish.

Proof Consider $g(t) = f(a + th), t \in (-1, 1)$, and h small. Then g has a local extremum at the point $t = 0$, i.e. at $g(0) = f(a)$. Hence we know by 1 variable calculus that the derivative there will be zero. But we realize that $g'(0)$ is the directional derivative of f at a in the direction h. But h is arbitrary, and $f'(a)$ is a linear function, hence we can multiply h by any scalar to obtain that the total derivative has to be zero to ensure that all directional derivatives vanish.

Definition Consider $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in Int(D)$. a is called a stationary or critical point if the total derivative $f'(a) = 0$ at that point. Note that all local extremum are critical points.

Saddle point If $f'(a) = 0$ but a is not a local extremum, then a is called a saddle point.

Hessian Test Aka second derivative test. Examine this for $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, D open, f is in C^2. Let $a \in D$ be a critical point. Let $A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial x \partial y}, C = \frac{\partial^2 f}{\partial y^2}$. Define $\Delta = AC - B^2$, the determinant of the Hessian matrix $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$. (1) If $\Delta < 0$, then a is a saddle point. (2) If $\Delta > 0$ and $A > 0$, then a is a local minimum. (3) If $\Delta > 0$ and $A < 0$, then a is a local maximum. (4) If $\Delta = 0$, then the test is inconclusive.

Hessian Test in n-variables Write the Taylor formula to second order for f in C^2, $f : D \subseteq \mathbb{R}^n \to \mathbb{R}, a \in Int(D)$. $f(a+h) = f(a) + f'(a)h + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)h_i h_j + ||h||^2 E_2(a, h)$. At a critical point $f'(a) = 0$. Hence, when h is sufficiently small, the second term would determine the sign of $f(a + h) - f(a)$. We form the Hessian matrix $(\frac{\partial^2 f(a)}{x_i x_j})_{i,j=1}^n$. Note that since f is in C^2, the mixed partials are equal, and the Hessian matrix is symmetric. Then (1) if all its eigenvalues of $H_f(a)$ are strictly positive, f has a relative minimum at a (2) if all its eigenvalues of $H_f(a)$ are strictly negative, f has a relative maximum at a (3) if it has some positive and some negative eigenvalues, then f is a saddle point at a. (4) If some eigenvalues are zero, we cannot tell.

About symmetric matrices Suppose A is a real symmetric $n \times n$ matrix. Then we associate with A the quadratic form $Q_A : \mathbb{R}^n \to \mathbb{R}$, a function of n variables. So for $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$, $Q_A(h) = \sum_{i,j} a_{ij} h_i h_j$. The quadratic form coresponding to the matrix A is called positive definite of $Q_A(h) > 0$ if $h \neq 0$. $Q_A(0) = 0$ obviously. Similarly, Q_A is negative definite of $Q_A < 0$ if $h \neq 0$.

Proposition: Symmetric matrices If Q_A is positive definite, then there is a positive number $M > 0$ such that $Q_A(h) \geq M||h||^2$ for every h.

Proof of proposition Multiply h by λ. Then, by inspection of the quadratic form $Q_A(\lambda h) = \lambda^2 Q_A(h)$. Hence we can re-write (for $h \neq 0$), Pick $\lambda = 1/||h||$. Hence $Q_A(h/||h||) = Q_A(h)/||h||^2$. But $h/||h||$ are the points on the unit sphere. Hence it suffices to find an M such that when x is on the unit sphere, then $Q_A(x) \leq M$. But the unit sphere is a compact set. Hence by compactness, such an M exists. Then the proposition follows.

Criterion for positive definiteness Let $A = (a_{ij})$ be a symmetric $n \times n$ real matix. Then Q_A is positive definite iff all its eigenvalues are strictly positive. It is negative definite if all its eigenvalues are negative. Recall that a symmetric real matrix has real eigenvalues.

Chapter 4

Week 4

4.1 Lecture 21 Apr 2014

Diagonalizing the Hessian Note that if A is diagonal, then the entries along the diagonal are the eigenvalues, and the quadratic form associated with it is $Q_A(h) = \lambda_1 h_1^2 + \ldots + \lambda_n h_n^2$. So the only way for $Q_A(h) > 0$ for **all** h is for $\lambda_1 \ldots \lambda_n > 0$.

Proof of Hessian Test in n-variables We prove (1), that f has a relative minimum at a if all the eigenvalues of the Hessian are strictly positive. (2) follows from (1) by multiplying the function by -1. We consider the Taylor formula for $f(a+h) = f(a) + f'(a)h + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)h_i h_j + ||h||^2 E_2(a,h)$. We note that the second derivative term in the Taylor formula is $\frac{1}{2}Q(a)$, where $Q(a)$ is the quadratic form corresponding to the Hessian at a. Taking into account that $f'(a) = 0$ since a is a critical point, we rewrite the Taylor expansion as: $f(a+h) - f(a) = \frac{1}{2}Q(a) + ||h||^2 E_2(a,h)$. We need to show that $f(a+h) - f(a)$ is dominated by $\frac{1}{2}Q(a)$. We assume that the eigenvalues of $H(a)$ are positive. Hence there is $M > 0$ such that $Q(h) \geq M||h||^2$. Hence we need to find $\delta > 0$ such that if $0 < ||h|| < \delta$, then $E_2(a,h) < M/4$. Hence $||h||^2 E_2(a,h) < \frac{M}{4}||h||^2$. Hence we have $\frac{1}{2}Q(h) \geq \frac{M}{2}||h||^2$, so $f(a+h) - f(a) > 0$ if h is small enough. Hence f has a local minimum at a.

Proof for 2 dimensions Define A, B and C as per theorem 9.7. We note that the Hessian eigenvalues λ_1, λ_2 satisfy $\lambda_1 + \lambda_2 = A + C, \lambda_1\lambda_2 = \Delta$. Note that if $\Delta < 0$, then λ_1, λ_2 have the opposite signs, and a is a saddle point. If $\Delta > 0$, then λ_1, λ_2 have the same sign. Also, $AC > B^2$, so A, C have the same sign. If $A > 0$, then $A + C > 0$ so $\lambda_1 + \lambda_2 > 0$, so $\lambda_1, \lambda_2 > 0$, so we have a local minimum at a. Repeat for $A < 0$ for local maximum.

Alternative criteria for positive definiteness Consider the corner matrices (square matrices including the top left hand corner). Then the matrix is positive definite if all the determnants are greater than 0. The matrix is negative definite if the determinants alternate between strictly negative and strictly positive.

Constraint-extrema and Lagrange Multipliers General problem: given $f : D \in \mathbb{R}^n \to \mathbb{R}$, D open, and constraints $g_1, \ldots, g_m : D \subseteq \mathbb{R}^{\to}\mathbb{R}, m < n$. Call the set $S = \{x \in D : g_1(x) = 0, \ldots, g_m(x) = 0\}$. A point $a \in S$ is a relative/local maximum of f at S if $f(a) > f(x)$ for all $x \in S \cap B_a(r)$ for some $r > 0$. Define a local minimum similarly.

Method of Lagrange Multipliers Consider the general problem described above. Let $f, g_1, \ldots, g_m$ be in C^1. Let $x_0 \in S$ be such that $\nabla g_1(x_0), \ldots, \nabla g_m(x_0)$ are (always) linearly independent. Then if x_0 is a local extremum of f in S, then $\nabla f(x_0)$ is a linear combination of $\nabla g_1(x_0) \ldots \nabla g_m(x_0)$. That is, there are $\lambda_1, \ldots, \lambda_m$ such that $\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \ldots + \lambda_m \nabla g_m(x_0)$. Note that if $m = 1$, then we just require that $\nabla g(x_0) \neq 0$ for linear independence.

Proposition If non-empty $F \subseteq \mathbb{R}^n$ is closed and $x_0 \in \mathbb{R}^n$, then there is a point in F closest to x_0.

4.2 Lecture 23 Apr 2014

Sketch of proof We consider a differentiable path passing through $x_0 \in S$ that is entirely contained in S. Let this path be $\alpha : I \to S$ differentiable with $\alpha(0) = x_0$. Consider the function $h(t) = f(\alpha(t))$ so that $h(0) = f(x_0)$. Then x_0 will also be a local extremum of $h(t)$. By 1 variable calculus, this means that the derivative will be zero at x_0. By the chain rule, we obtain that $h'(t) = \nabla f(x_0) \cdot \alpha'(0)$, so $\nabla f(x_0) \perp \alpha'(0)$. Now since the path lies on S, it must satisfy the constraints $g_i(\alpha(t)) = 0$. Differentiating this and using the chain rule, we obtain that $\nabla g_i(x_0) \cdot \alpha'(0) = 0$. Hence we have that $\nabla g_1(x_0) \perp \alpha'(0)$. Combining this with other constraint conditions, we have that $\alpha'(0) \perp < \nabla g_1(x_0), \ldots, \nabla g_m(x_0) >$, the subspace spanned by the m gradient vectors. It remains to show that any vector orthogonal to the subspace is a velocity vector $\alpha'(0)$ for some path

α. (Need advanced calculus for this. This is true by the implicit function theorem.) Then $\nabla f(x_0)$ will be orthogonal to any vector orthogonal to the subspace $< \nabla g_1(x_0), \ldots, \nabla g_m(x_0) >$. Hence $\nabla f(x_0)$ is in the subspace, and is a linear combination of the vectors spanning the subspace.

Implicit function theorem Suppose a function has continuous partial derivatives. Assume that the gradient is non-zero at (a_0, b_0, c_0). This means that at least one of the partial derivatives is not zero. WLOG, assume that the partial derivaitve with respect to z is non-zero. Then around $(x, y) = (a_0, b_0)$, $g(x, y, z) = 0$ is exactly the graph of a C^1 function $h(x, y)$ where $h(a_0, b_0) = c_0$.

Riemann integration Consider a closed n-dimensional box $B = [a, b] = [a_1, b_1] \times \ldots \times [a_n, n_n] \in \mathbb{R}^n$. The volume of the box is defined to be the product $\prod_{i=1}^n (b_i - a_i)$. The regular partition of the box consists of subdividing each side $[a_i, b_i]$, to form the sequence $a_i = x_0^i < x_1^i < \ldots < x_n^i < b_i = x_{n+1}^i$, and considering the boxes $\prod_{i=1}^n [x_{k_i}^i, x_{k_i+1}^i]$. Write $P = \{S_j\}_{j=1}^k$ for such a partition. A refinement of the partition P is such that the boxes in P are entirely contained in the refinement. Note that any two partitions P_1, P_2 have a common refinement P. Assume we have a scalar function $f : B \to \mathbb{R}$ that is bounded. Take the partition $P = \{S_j\}_{j=1}^k$ to be a partition of B. For each box, take the supermum $\beta_j = \sup f(S_j)$ and $\alpha_j = \inf f(S_j)$. Define the upper sum $U(f, P) = \sum_{j=1}^k vol(S_j)\beta_j$ and the lower sum $L(f, P) = \sum_{j=1}^k vol(S_j)\alpha_j$. Note that $L(f, P) \leq U(f, P)$. If P' is a refinement of P, then $L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$. Note that if $\alpha = \sup f(B)$ and $\beta = \inf f(B)$, then the upper and lower integrals are bounded below by $vol(B)\alpha$ and bounded above by $vol(B)\beta$. Define the lower integral to be $\underline{I}(f) = \sup_P L(f, P)$ and upper integral $\bar{I}(f) = \inf_P U(f, P)$. Notet that $\underline{I}(f) \leq \bar{I}(f)$. A function is integrable if $\underline{I}(f) = \bar{I}(f)$. Equivalently, this means that given $\epsilon > 0$, there is P such that $U(f, P) - L(f, P) < \epsilon$. If the integral exists, we write $\int_B f$ to be the integral of the function f over B.

4.3 Lecture 25 Apr 2014

Step function f is a step function in B if f is bounded and there is a partition $P = \{P_j\}_{j=1}^k$ such that f is constant with some value c_j on P_j. Then f is integrable and the value of this integral is $\sum_{j=1}^k c_j vol(P_j)$. Note that we do not care what happens on the boundary of each sub-partition.

Proof We create a refinement of the partition P by introducing divisions arbitrarily near the boundaries of each earlier partition. We then prove that for all $\epsilon > 0$ we can form a refinement of the given partition P_ϵ such that $U(f, P_\epsilon)$ and $L(f, P_\epsilon)$ are within ϵ of the sum $\sum_{j=1}^k c_j vol(P_j)$. Hence this discrete sum must be the common value of the lower and upper integrals.

Theorem Let f be continuous on B. Then f is integrable. That is, any continuous function is integrable.

Proof This proof is based on Uniform Continuity aka Small Span Theorem. Take any ϵ and find the δ using uniform continuity. ???

Uniform Continuity/Small Span Theorem If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^k$ is continuous and its domain D is compact, then f is uniformly continuous: $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, y \in D$, $||x - y|| < \delta \implies ||f(x) - f(y)|| \leq \epsilon$.

Proof We fix $\epsilon > 0$. We want to find a $\delta > 0$ that works for every set of points $x, y \in D$. By continuity, given $x \in D$, there is $r_x > 0$ such that $y \in B_{r_x}(x) \implies ||f(y) - f(x)|| < \epsilon/2$. Hence if $y, z \in B_{r_x}(x)$ then $f(y) - f(z)|| < \epsilon$. Call $s_x = \frac{1}{2} r_x$. Then D is covered by union of open balls $\cup\{B_x(s_x) : x \in D\}$, so by compactness, it is contained in finitely many of them, say m many. We pick δ to be the minimum of s_x in the finitely many sets. Hence if $x, y \in D$ and if $||x - y|| < \delta$, then for some i, $||x - x_i|| < s_{x_i}$ for some $1 \leq i \leq m$. So $||y - x_i|| < 2s_{x_i} = r_{x_i}$ by the triangular inequality. Hence both $x, y \in B_{r_{x_i}}(x_i)$ so $||f(x) - f(y)|| < \epsilon$.

Definition: Content zero A bounded set $A \subseteq \mathbb{R}^n$ has content 0 if for every positive ϵ, there are closed boxes $Q_1, \ldots Q_m$ such that A is contained in the union of the boxes, and the total volume of the boxes $vol(Q_1,) + \ldots + vol(Q_m) < \epsilon$. Observation: If A has content zero and $B \subseteq A$, then B also has content zero. Also, if we have finitely many sets with content zero, their union also has content zero. This is because we can cover each with set with boxes with total volume smaller than ϵ/n, if there are n such sets, and the total volume will be less than ϵ. Also note that if we have a path $\phi : B \to \mathbb{R}$, where $B \subseteq \mathbb{R}^n$ is a closed box and ϕ is continuous, then the graph of ϕ is a set in $\mathbb{R}^{n+1}, graph(\phi) = \{(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in B, \phi(x_1, \ldots, x_n) = y\}$. Then the graph of ϕ has content zero. Also, if A has content zero, so does its closure. Prove this by $A \subseteq Q_1 \cup \ldots \cup Q_m \implies \bar{A} \subseteq \overline{Q_1 \cup \ldots \cup Q_m} = \bar{Q_1} \cup \ldots \cup \bar{Q_m} = Q_1 \cup \ldots \cup Q_m$. Note that even though the real interval $[0, 1]$ does not have content zero in the real line, it has content zero if considered in $\mathbb{R}^2$, since it is just a straight line and can be covered by rectangles that are arbitrarily small.

Theorem If $f : B \to \mathbb{R}$ is bounded, and $B \subseteq \mathbb{R}^n$, and f is continuous except at every point in the set of content zero (i.e. the set of discontinuities has content zero), then f is integrable.

Definition: Jordan Measureable A bounded set $A \subseteq \mathbb{R}^n$ is Jordan measureable (hence is continuous) if $\partial(A)$ has content zero. Let $f : A \to \mathbb{R}$ be a bounded function which is continuous in the interior of a Jordan measureable set. Then we can define the integral of f by considering the integral of another function $\int_B \bar{f}$, where B is a closed box containing A (which is possible since A is bounded), and where $\bar{f}$ extends f to B such that $f(x) = 0$ if $x \in B, x \notin A$. Now we note that $\bar{f}$ is continuous in A and outside A (but in B), but is not continuous along the boundary. But the set of boundary points has content zero, hence $\bar{f}$ is integrable. Note that we can choose any such B, since the values outside A does not contribute to the overall value of the integral.

Definition: Volume of a Jordan Measureable set If A is Jordan measureable, define $vol(A) = \int_A 1$.

Fubini's theorem Consider $n = 2$ first. Let f be a bounded, integrable function defined on a box $B = [a_1, b_1] \times [a_2, b_2]$. Let $A(x) = \int_{a_2}^{b_2} f(x, y)dy, x \in [a_1, b_1]$ exist as an integral of one variable. Assume $A(x)$ is integrable on $[a_1, b_1]$. Then $\iint_B f(x, y)dxdy = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} f(x, y)dy\right] dx$. Note that we can switch the order of integration too.

Fubini's theorem for continuous functions If $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous, then $A(x)$ exists and is integrable. We note that $f(B)$ is automatically bounded since B is a closed box and f is continuous. We can hence use the iterated integral form for the double integral. Note that if a function is integrable, it does not necessarily mean it is continuous. It can be discontinuous on a set of content zero.

Fubini's theorem for n dimensions If f is continuous on $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$, then we can consider the integral of f over B using the iterated integrals in each dimension. Note that the order of integration does not matter.

Chapter 5

Week 5

5.1 Lecture 28 Apr 2014

Alternative formulation of uniform continuity Consider $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, consider a subset of its domain $S \subseteq D$. Let f on D be bounded. Hence there exists the sup and inf $\beta = \sup f(S), \alpha = \inf f(S)$, with $\alpha \leq \beta$. Define the span or oscillation of f on S to be $osc(f, S) = \beta - \alpha$. The diameter of S is $diam(S) = \sup\{||x - u|| : x, y \in S\}$. To say that f is uniformly continuous on S is to say that $\forall \epsilon > 0, \exists \delta > 0$ so that for any subset $S \in D$, if $diam(S) < \delta$, then $osc(f, S) < \epsilon$. If f is continuous and D is compact, then f is uniformly continuous on D.

Theorem If $B \subseteq \mathbb{R}^n$ is a box and $f : B \to \mathbb{R}$ is continuous, then f is integrable.

Proof f is uniformly continuous on B, since the closed bx is bounded and closed, hence it is a compact set. Fix $\epsilon > 0$ and let $\delta > 0$ be such that $||x - y|| < \delta$ then $|f(x) - f(y)| < \epsilon$ for $x, y \in B$. Now we need to find a fine enough partition such that the difference between the upper and lower sums is less than ϵ. Let $P = \{P_j\}_{j=1}^k$ be a partition of the box B such that the $diam(P_j) < \delta$. Then the oscillation of the function on each box $osc(f, P_j) \leq \epsilon$. Call $\beta_j = \sup f(P_j)$ and $\alpha_j = \inf f(P_j)$, then the difference between the upper and lower sum if by definition $\sum_j (\beta_j - \alpha_j) vol(P_j) = \sum_j osc(f, P_j) vol(P_j) \leq \epsilon \times vol(B)$. Now the volume of B is a fixed number, and epsilon is arbitrary, hence we can make the difference arbitrarily small too.

Stronger version of theorem We do not require that the function is continuous everywhere on the closed box. Let $B \subseteq \mathbb{R}^n$ be a closed box, $f : B \to \mathbb{R}$ be bounded and continuous except at a set of points with content zero. Then f is integrable on B.

Sketch of Proof Enclose the discontinuous points in a collection of rectangles in a partition. Now in the other rectangles, the function is continuous, and hence is uniformly continuous on those rectangles. Hence we can find a finer partition for the continuous rectangles to find that its oscillation can be made arbitrarily small. Now we can make the collection of discontinuous rectangles have arbitrarily small volume.

Even stronger theorem We can characterize which functions are integrable on a box. A set is call null if it can be covered by an infinite sequence of rectangles of content zero. If a function has discontinuities that is Lebesque null, then it is integrable.

Graph of continuous functions Let $\phi : B = [a, b] \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous. Then the graph of this function $graph(\phi) \subseteq \mathbb{R}^{n+1}$ has content zero.

Proof Consider $n = 1$ for simplicity. Choose an $\epsilon > 0$. Partition the interval $[a, b]$ into $\{P_j\}_{j=1}^k$ small enough such that on P_j the oscillation of $\phi \leq \epsilon$. This is possible by uniform continuity. Hence, we can enclose the graph of that small subpartition in a box B_j of volume $vol(P_j) \cdot \epsilon$. Hence the whole graph of ϕ is contained in finitely many boxes of total volume $\leq \sum_{j=1}^k vol(P_j)\epsilon = \epsilon \cdot vol(B)$. Hence we can make this volume arbitrarily small.

Fubini's theorem proof (refer to previous day's notes) Partition $[a_1, b_1]$ into $\{S_j\}_{j=1}^m$ and $[a_2, b_2]$ into $\{T_k\}_{k=1}^n$. This gives a partition $P = \{S_j \times T_k\}$ of B. Now we define the step functions $s, t : B \to \mathbb{R}$ by $s = \inf(f(S_j \times T_k))$ on $Int(S_j \times T_k)$, and t to be the supremum of the same. We note that $s \leq f \leq t$. Define the lower sum of f on P to be $L(f, P) = \int_B s$, and the upper sum if $U(f, P) = \int_B t$. It is easy to check that the Fubini theorem holds for finite step functions: $\sum_{1 \leq j \leq m, 1 \leq k \leq n} a_{jk} = \sum_{1 \leq j \leq m} \left(\sum_{1 \leq k \leq n} a_{jk} \right)$. Hence we have that $s(x) = \int_{a_2}^{b_2} s(x, y) dy \leq \int_{a_2}^{b_2} f(x, y) dy = A(x) \leq \int_{a_2}^{b_2} t(x, y) dy = t(x)$. Hence $\iint_B s = \int_{a_1}^{b_1} [\int_{a_2}^{b_2} s(x, y) dy] dx \leq \int_{a_1}^{b_1} [\int_{a_2}^{b_2} f(x, y) dy] dx \leq \int_{a_1}^{b_1} [\int_{a_2}^{b_2} t(x, y) dy] dx = \iint_B t$. But P was an arbitrary partition. Hence f is integrable over B, and is equal to the iterated integral.

Integration over special regions Define a region of type I (in $\mathbb{R}^2$) to be contained within the graph of two functions: $\psi(x), \phi(x)$. We can write this as $\{(x, y) : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\}$ where ψ, ϕ are continuous with $\phi(x) \leq \psi(x)$. A region of type II is analogous to type I, where we contain the region bounded by $\phi(y)$ and $\psi(y)$ instead. A region of type III is of both type I and II, which can be viewed as between two functions in the x direction or y direction. E.g. of type III domains: boxes and spheres. Now we can talk about the integral of the function on each of these domains, since the boundaries are graphs of functions, and hence the boundary has content zero.

Integral of type I domain Let D be of type I, f continuous on D. Then the integral of f on D $\iint_D f = \int_a^b [\int_{\phi(x)}^{\psi(x)} f(x, y)dy]dx$. This holds for type II domains too, except that we reverse the order of integration. Domains of type III can be done in either order.

3 dimension domains A type I domain is the set of all triples $\{(x, y, z) \in \mathbb{R}^3 : a_1 \leq x \leq b_1, \phi(x) \leq y \leq \psi(x), \eta(x, y) \leq z \leq \theta(x, y)\}$ with ϕ, ψ, η, θ continuous. Define the type II domain by switching the conditions of x and y. Then the integral is $\iiint_D f = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} \left[\int_{\eta(x,y)}^{\theta(x,y)} f(x, y, z)dz\right] dy\right] dz$.

5.2 30 Apr 2014 Midterm Review

Question: How do you know whether an extrema on a constrained function is a local min or a max? Do Lagrange multipliers give saddle points?

5.3 Recitation 01 May 2014

Examples Consider $A = (0, 1) \cup \{2\} \subseteq \mathbb{R}$. What are the interior points and boundary points of A? Is A open or closed?

Solution The interior points of are $Int(A) = (0, 1)$. Consider a point $a \in (0, 1)$. Then $a \in A$. Also, consider the open neighbourhood $B_a(r)$ around a. Since $0 < a < 1$, we choose $r = \min(1 - a, a)/2$. Then the open neighbourhood will be contained entirely in A. Hence a is an interior point of A. The boundary points of A are $0, 1, 2$. Note that every open ball centered at each of these points will contain points that are in A and not in A. We have that $B_0(r) \cap A \neq \phi$ and $B_0(r) \cap A^c \neq \phi$.

Example 2 Consider $\mathbb{Q} \subseteq \mathbb{R}$. Same question as above. Fact: $\mathbb{Q}$ is dense in $\mathbb{R}$. This means that for all $x \in \mathbb{R}, \forall \epsilon > 0, \exists q \in \mathbb{Q}$ such that $|q - x|\epsilon$. This means that there is always a rational number arbitrarily close to any real number. In the opposite direction, $\forall q \in \mathbb{Q}, \forall \epsilon > 0, \exists x \in \mathbb{R} \backslash \mathbb{Q} : |q - x| < \epsilon$.

Solution Let $q \in \mathbb{Q}$, any interval around q has to contain q. But the interval has to contain point that are irrational too. Hence that point q is a boundary point. Hence it means that $\mathbb{Q}$ is contained in its boundary $\partial\mathbb{Q}$. Now we need to check an arbitrary irrational point $x \in \mathbb{R} \backslash \mathbb{Q}$. x is also a boundary point. Hence the boundary is the whole of the real line. Q is neither open nor closed.

Example 3 Let $f(x, y) = x^2 + y^2$. Show that f is differentiable. Using an easy criterion, f is differentiable if its partial derivatives exist and are continuous. $D_1 f = 2x, D_2 f = 2y$, which are clearly continuous. From the definition, we have that f is differentiable if there exists a linear transformation $L(x, y)$ with $L : \mathbb{R}^2 \to \mathbb{R}$ such that $\lim_{u \to 0} \frac{||f(x+u_1, y+u_2) - f(x,y) - L(u_1, u_2)||}{||u||} = 0$. But we know that this is going to be the gradient $\nabla f(x, y) = (2x, 2y)$. Writing out the limit,

$$\begin{aligned}
Limit &= \lim_{u \to 0} \frac{||(x + u_1)^2 + (y + u_2)^2 - x^2 - y^2 - 2xu_1 - 2yu_2||}{||u||} \\
&= \lim_{u \to 0} \frac{u_1^2 + u_2^2}{\sqrt{u_1^2 + u_2^2}} \\
&= \lim_{u \to 0} ||u|| \\
&=0
\end{aligned}$$

Example 4 Find the global extrema of $f(x, y, z) = 3x + 3y + 8z$ subject to the constraints $g_1(x, y, z) = x^2 + z^2 - 1 = 0$ and $g_2(x, y, z) = y^2 + z^2 - 1 = 0$. We first note that f is continuous as it is a polynomial. Also, the domain is bounded and closed, because a single cylinder is a closed set, and the intersection of two closed sets is still closed. The domain is bounded because we can rearrange the constraints to be $y^2 = 1 - z^2 \leq 1$. Hence $|y| \leq 1$. Also, $x^2 = 1 - z^2 \leq 1$. Hence $|x| \leq 1$. Also $z^2 = 1 - x^2 \leq 1$ So $|z| \leq 1$. Since $|x|, |y|, |z| \leq 1$, the domain is bounded. Hence the constrained extrema exists. We first find the gradient of the constraints: $\nabla g_1 = (2x, 0, 2z), \nabla g_2 = (0, 2y, 2z)$. We note that as long as $(x, y, z) \neq (0, 0, 0)$, then the gradient vectors are linearly independent, since they cannot be a linear multiple of the other. But we know that the point $(0, 0, 0)$ does not satisfy the constraints. Hence we have that $(x, y, z) \neq (0, 0, 0)$, and the constraint gradients are linearly independent.

5.4 Lecture 02 May 2014

Volume below graph of two-dimensional function Let $f \geq 0$ be a non-negative function defined on $B = [a_1, b_1] \times [a_2, b_2]$. Consider the set $F = \{(x, y, z) : (x, y) \in B, 0 \leq z \leq f(x, y)\}$. The boundary of this set has content zero, hence it is a Jordan measureable set. By definition, the volume of F: $Vol(F) = \iiint_F 1$. We want to show that $Vol(F) = \iint_B f(x, y)dxdy$. By Fubini's theorem, $Vol(F) = \iiint_F 1dxdydz = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} \left[\int_0^{f(x,y)} 1dz\right] dy\right] dx = \iint_B f(x, y)dxdy$.

Example: Volume of ellipsoid Let $F = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$, $a, b, c > 0$. The projection of this ellipsoid to the xy plane is an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. Applying the Fubini theorem, we have $Vol(F) = \iiint_F 1dxdydz = \int_{-a}^{a} \left[\int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \left[\int_{-c\sqrt{1-(x/a)^2-(y/b)^2}}^{c\sqrt{1-(x/a)^2-(y/b)^2}} 1dz\right] dy\right] dx = \frac{4\pi}{3}abc$.

Example: Volume of tetrahedron Consider tetrahedron with corner at origin, and points at $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Then $T = \{(x, y, z) : x, y, z \geq 0, x + y + z \leq 1\}$. This is effectively the solid under the function $f(x, y) = 1 - x - y$ on the domain $\{(x, y) : x, y \geq 0, x + y \leq 1\}$. Hence the volume is $Vol(T) = \int_0^1 \int_0^{1-x} (1 - x - y)dydx$.

Solid of revolution Calculates the volume of the solid of revolution. Consider a type I region in $\mathbb{R}^2$ bounded below by $\phi(x)$ and above by $\psi(x)$, and with x bounded between $[a, b]$. Rotate this region around the x-axis. Consider first the solid of revolution obtained by rotating the graph under the function $\psi(x)$: $\{(x, y) : a \leq x \leq b, y \leq \psi(x)\}$ around the x-axis. Call this S_1. Similarly, rotate the graph of the smaller function $\phi(x)$ and call its volume S_2. Then the required volume is the difference of the two volumes $S = S_1 - S_2$. Note that we can write $S_1 = \{(x, y, z) : a \leq x \leq b, -\psi(x) \leq y \leq \psi(x), -\sqrt{\psi(x)^2 - y^2} \leq z \leq \sqrt{\psi(x)^2 - y^2}\}$. We can apply Fubini's theorem to this to obtain: $Vol(S_1) = \int_a^b \int_{-\psi(x)}^{\psi(x)} \int_{-\sqrt{\psi(x)^2-y^2}}^{\sqrt{\psi(x)^2-y^2}} 1dzdydx = \int_a^b \pi\psi(x)^2dx$. Similarly $Vol(S_2) \int_a^b \pi\phi(x)^2dx$. Hence $Vol(S) = Vol(S_1) - Vol(S_2) = \int_a^b \pi(\psi(x)^2 - \phi(x)^2)dx$.

Centroid of Jordan Measureable set The centroid of a Jordan measureable set $Q \subseteq \mathbb{R}^2$ is the point $(\bar{x_Q}, \bar{y_Q}) = (\frac{\iint_Q xdxdy}{area(Q)}, \frac{\iint_Q ydxdy}{area(Q)})$. It is the average x and y coordinate. The $area(Q) = \iint_Q 1dxdy = A$. If $Q = \{(x, y) : x \in [a, b], \phi(x) \leq y \leq \psi(x)\}$. Then $\bar{y} = \frac{\int_a^b \int_{\phi(x)}^{\psi(x)} ydydx}{A} = \frac{\int_a^b (\psi(x)^2 - \phi(x)^2)dy}{2A}$. Then $Vol(S) = 2\pi\bar{y}A$.

Line Integral Recall that a continuous path or parametrized curve is the continuous function $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$. A differentiable path means that $\alpha'(t)$ exists at every t on the domain $[a, b]$. α is in C^1 if $\alpha'(t)$ is continuous. Call a path piecewise C^1 if there is a partition of $[a, b]$ such that α is C^1 on each subpartition. α is simple, if α is 1-1. Hence it does not cut itself. A closed path is a path such that $\alpha(a) = \alpha(b)$. A closed path is simple.

Rectifiable Path and Length Cosnider a path $\alpha : [a, b] \to \mathbb{R}^n$ continuous, and consider a partition $P = \{t_0 = a, t_1, \ldots, t_{m-1}, t_m = b\}$. Consider the polygonal path $\alpha(a) \to \alpha(t_1) \to \alpha(t_2) \to \ldots$. This path has length $L(P) = \sum_{i=0}^{m-1} ||\alpha(t_{i+1}) - \alpha(t_i)||$. α is rectifiable if all $L(P)$ are bounded. That is $\exists M > 0 \forall P, L(P) \leq M$. Then define the length of α to be $L(\alpha) = \sup_P L(P)$. Not every continuous path is rectifiable.

Theorem If α is piecewise C^1, then α is rectifiable and $L(\alpha) = \int_a^b ||\alpha'(t)||dt = \int_a^b ||v(t)||dt$.

Chapter 6

Week 6

6.1 Lecture 05 May 2014

Example: Length of an ellipse Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Parametrize this curve using $\alpha(t) = (a\cos t, b\sin t), 0 \le t \le 2\pi$. The derivative is $\alpha'(t) = (-a\sin t, b\cos t)$, so $||\alpha'(t)|| = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$. By symmetry of the curve, we can write the length as $4\int_0^{\pi/2}\sqrt{a^2\cos^2 + b^2\sin^2 t}dt$. This is an elliptic integral.

Arclength function Suppose we have a piecewise C^1 path $\alpha : [a,b] \to \mathbb{R}^n$. Consider a $t \in [a,b]$. Now we define $s(t) = \int_a^t ||\alpha'(u)du$, the length of the path between a and the arbitrary point t along the path. Clearly, $s(a) = 0$, $s(b) = L(\alpha)$. Also, $s(t)$ is increasing. If the derivative $\alpha'(t) \neq 0$ along the path, then $s(t)$ is strictly increasing. Now by the fundamental theorem of calculus, $s'(t) = ||\alpha'(t)|| = v(t)$. $s(t)$ is called the arc-length function of α.

Line integrals of vector fields Consider a piecewise C^1 path $\alpha : [a,b] \to \mathbb{R}^n$. Let the curve of this path $C = \alpha([a,b])$ be contained in $D \subseteq \mathbb{R}^n$. Also consider a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ to be a bounded vector field. Then the line integral of f over α is $\int f \cdot d\alpha \equiv \int_a^b f(\alpha(t)) \cdot \alpha'(t)dt$. We can also write this as $\int_C f \cdot d\alpha$, but note that the integral depends on α, not just C.

Geometric motivation Suppose $\alpha(t)$ is the position of a moving particle at time t. Consider the force field $F : \mathbb{R}^n \to \mathbb{R}^n$. Then the work done by the field on the particle is the line integral of F along α: $W = \int F(\alpha(t)) \cdot \alpha'(t)dt$.

Parameter change First consider a path $\alpha : [a,b] \to \mathbb{R}^n$. Consider a change of parameters: $u : [c,d] \to [a,b]$ which is bijective, C^1 and has non-zero derivative $u'(t) \neq 0$. The last condition (along with continuity of the derivative) means that u' is always positive or negative on $[c,d]$. We call this change of parameters orientation preserving if $u' > 0$ and orientation reversing if $u' < 0$. Consider the composition $\beta = \alpha \circ u$. Then $\beta(t) = \alpha(u(t)), t \in [c,d]$.

Example Consider the parameter change that maps $a \to b$ and $b \to a$. $u(t) = -t + a + b$. Since $u'(t) = -1 < 0$, it is an orientation reversing parameter change.

Equivalent path Call two paths equivalent, or $\alpha \sim \beta$ if $\beta = \alpha \circ u$ for some orientation preserving u. Note that if u is orientation preserving, then u^{-1} is also orientation preserving. Equivalence is transitive, so if $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

Theorem: Integral invariance under parameter change Assume α, β are piecewise C^1 and $\alpha \sim \beta$. Then for any continuous f: $\int f \cdot d\alpha = \int f \cdot d\beta$.

Proof This is a straightforward application of the chain rule. Assume that α is C^1. If α is only piecewise C^1, we just break the integral into finitely many partitions that are C^1 on each. Now $\beta(t) = \alpha(u(t))$ with orientation preserving u. By the chain rule $\beta'(t) = \alpha'(u(t))u'(t)$. Note that $\alpha'(u(t))$ is a vector while $u'(t)$ is a real number. So the integral with respect to β is, by definition, $\int_c^d f \cdot d\beta = \int_c^d f'(\beta(t)) \cdot \beta'(t)dt = \int_c^d f(\alpha(u(t))) \cdot \alpha'(u(t))u'(t)dt$. Now we can make the change of variable $v = u(t), dv = u'(t)dt$. Then we can write the integral as $\int_{u(d)=a}^{u(c)=b} f(\alpha(v)) \cdot \alpha'(v)dv = \int f \cdot d\alpha$. Note that if u is orientation reversing, then we have that $\int f \cdot d\alpha = -\int f \cdot d\beta$.

Example Take $\alpha \in C^1$, $s(t) = \int_a^t ||\alpha'(p)||dp = \int_a^t v(p)dp, t \in [a,b]$, where p is just a dummy variable. Then $s : [a,b] \to [0, L(\alpha)]$. If $v(p) \neq 0$, then the integral is strictly increasing. Now consider the change of parameter $u = s^{-1} : [0, L(\alpha)] \to [a,b]$ is an orientation preserving reparametrization of α. Then the domain of the composition is $[0, L(\alpha)]$.

Line integrals of scalar fields with respect to arclength Consider a path $\alpha : [a, b] \to \mathbb{R}^n$ that is piecewise C^1. Assume that the curve traced by the path is contained in $D \subseteq \mathbb{R}^n$, and consider a scalar field $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. Then the integral with respect to arclength is defined to be $\int_C f ds = \int_\alpha f ds = \int_a^b f(\alpha(t))||\alpha'(t)||dt$, since the differential arclength is $ds = ||\alpha'(t)||dt$. If $\alpha \sim \beta$ or $\alpha \sim -\beta$, then the integral has the same value for both paths.

Geometric motivation Consider $\mathbb{R}^3$. Consider a thin wire parametrized by α. Let the linear density of the wire be $f(x, y, z)$. Then the mass of a small piece of wire is given by $dm = f(\alpha(t))||\alpha'(t)||dt$. Hence the integral with respect to arclength represents the total length of the wire. If the total mass is called M, then the center of mass of the wire is the point $(\bar{x}, \bar{y}, \bar{z})$, given by $\bar{x} = \frac{\int_C f(\alpha(t))xds}{M}$ and so on. When $f = 1$, then the center of mass is called the centroid of C.

Connecting the two types of integrals Consider the path α as before, $\alpha'(t) \neq 0$. We define the unit tangent vector $T(t) = \frac{\alpha'(t)}{||\alpha'(t)||}$. Let F be a vector field. Consider the scalar field $g(\alpha(t)) = f(\alpha(t)) \cdot T(t)$ which is the projection of $f(\alpha(t))$ along $T(t)$. Then we have that $\int_C gds = \int g(\alpha(t))||\alpha'(t)||dt = \int f(\alpha(t)) \cdot T(t)||\alpha'(t)||dt = \int f(\alpha(t)) \cdot \alpha'(t)dt = \int f \cdot d\alpha$.

6.2 Lecture 07 May 2014

Second Fundamental Theorem of (1D) Calculus Recall that $\int_a^b g'(t)dt = g(b) - g(a)$ for $g \in C^1$.

Second FTC for line integrals Given $\alpha : [a, b] \to \mathbb{R}^n$, α being piecewise C^1. Let $\alpha([a, b]) \subseteq D$ be open. Also let $g : D \subseteq \mathbb{R}^n \to \mathbb{R}$. Then. $\int \nabla g \cdot d\alpha = g(\alpha(b)) - g(\alpha(a))$. Note that this line integral does not depend on the path, but only on the end-points. This is not true when g is any scalar field; it must be a gradient. This reduces to the 1D version when $n = 1$ and when α is the identity. Also note that if α is a closed path, such that $\alpha(a) = \alpha(b)$, then the line integral is zero. Use $\oint_C \nabla g \cdot d\alpha = 0$ as the notation for a closed path integral.

Proof Assume WLOG that α is continuously differentiable. If not, we can just treat each finite piece separately. Consider the function $h(t) = g(\alpha(t)), t \in [a, b]$. Use the chain rule to obtain $h'(t) = \nabla g(\alpha(t)) \cdot \alpha'(t)$. Now $h'(t)$ is continuous since g and $\alpha(t)$ are continuous. Hence by the 1D FTC, the integral of $\int \nabla g \cdot d\alpha = \int_a^b \nabla g(\alpha(t)) \cdot \alpha'(t)dt = \int_a^b h'(t)dt = h(b) - h(a) = g(\alpha(b)) - g(\alpha(a))$.

Example Compute $\int f \cdot d\alpha$, where $f(x, y, z) = (2xy + z, x^2, x)$ along the path $\alpha(t) = (e^t, e^{2t}, t^2), 0 \leq t \leq 2$. We note that f is the gradient of ϕ, $\phi(x, y, z) = x^2y + xz$. Hence the integral is simply $\phi(\alpha(2)) - \phi(\alpha(0))$.

First FTC (1D) Recall that if $f \in C^1$ on $[a, b]$, then for $x \in [a, b]$, $(\int_a^x f)' = f$.

First FTC for line integrals Suppose we have $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ continuous with D open. Suppose that for every piecewise C^1 path included in D, the line integral of f along that path depends only on the endpoints, then f is the gradient of some $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi \in C^1$.

Proof postponed for now

Connectedness Consider an open set $U \subseteq \mathbb{R}^n$. We consider two points $x, y \in U$ to be equivalent: $x \sim y$ iff there exists a piecewise C^1 path contained in U from x to y. Now we note that $x \sim x$, and if $x \sim y$, then $y \sim x$. Also, if $x \sim y$ and $y \sim z$, then $x \sim z$. The equivalence class which is denoted by $[x]_\sim = \{y : x \sim y\}$, the set of all points equivalent to x, is called the connected component of x in U. Then U, which is any open set, is the disjoint union of its connected components. We note that $[x]_\sim$ is open. This is because U is open, hence we have an open ball around any point y contained entirely in U. Then if $x \sim y$, then we can pick a $z \in B_y(r)$, then connect the point z to the point y by a straight line, and we have a piecewise C^1 path connecting x to z. Hence $x \sim z$, and $z \in [x]_\sim$. We call U connected if it has only 1 connected component. This means that any point in U can be connected to any other point by a piecewise C^1 path.

Theorems on Connectness The following are equivalent: (1) U is connected. (2) Any 2 points in U can be connected by a continuous path contained entirely in U. (3) Any 2 points in U can be connected by a polygonal path parallel to the axes (i.e. can connect the points with a collection of horizontal or vertical paths in 2D, for instance). (4) U cannot be decomposed into two disjoint non-empty open sets.

Proof of the First FTC for line integrals We can assume, working separately on each connected component of D, that D is connected. Then we can take any point $x_0 \in D$ and define for each other $x \in D$, the following function $\phi(x) = \int f \cdot d\alpha$, where α is a piecewise C^1 path lying completely in D from x_0 to x. Now $\phi(x)$ is well defined, since the integral of f along any path does not depend on the path. Now we want to show that $\nabla\phi = f$. Hence we have to show that $\frac{\partial\phi}{\partial x_i} = f_i$. Now consider the straight line connecting x and $x + he_i, e_i$ being the ith basis vector, with h small enough so that the line is contained entirely in D. Now we take the path $\alpha(t) = x + t(he_i), t \in [0,1]$. Then $\phi(x + he_i) - \phi(x) = \int f \cdot d\alpha = \int_0^1 f'(\alpha(t)) \cdot \alpha'(t)dt = \int_0^1 h[f(x + the_i) \cdot e_i dt$. Hence dividing by h on both sides, $\frac{\phi(x+he_i)-\phi(x)}{h} = \int_0^1 f(x + the_i) \cdot e_i dt = \int_0^1 f_i(x + the_i)dt$. We now make the change of variable $u = th, du = hdt$, hence the RHS becomes $\frac{1}{h}\int_0^h f_i(x + ue_i)du = \frac{1}{h}(g(h) - g(0)$, where $g(y) = \int_0^y f_i(x + ue_i)du$. Now we take $h \to 0$. Then the LHS becomes $\frac{\partial\phi(x)}{\partial x_i} = g'(0) = f_i(x)$ by the 1D FTC.

Corollary Suppose we have a vector field $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open, f continuous. Then the following are equivalent. (1) f is the gradient of some $\phi : D \subseteq \mathbb{R}^n \to \mathbb{R}$. (2) Line integrals of f over paths in D depend only on the endpoints, (3) every $\oint f \cdot d\alpha = 0$. We call f a conservative field if these happen, and we call ϕ a potential function for f.

Proposition $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open and connected, f continuous. Then ϕ is uniquely determined up to a constant. Note it is important that D is connected. If D is not connected, we can defined f separately on each disjoint domain, and we can obtain a $\nabla\phi = 0$ for each set, but with ϕ having a different constant value for each disjoint set.

Proof It will suffice to show that if $\phi \in C^1$, and $\nabla\phi = 0$, then ϕ is a constant. Pick a path α between two points. Then $\phi(\alpha(t)) - \phi(\alpha(b)) = \int \nabla\phi \cdot d\alpha = 0$. Hence it means that $\phi(\alpha(a)) = \phi(\alpha(b))$, and $\phi(x) = \phi(y)$.

6.3 Lecture 09 May 2014

Conservative fields Consider a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, f continuous, D open. The following are equivalent: (1) f is conservative, (2) $f = \nabla\phi, \phi : D \subseteq \mathbb{R}^n \to \mathbb{R}$, (3) Line integrals of f over paths in D depend only on endpoints, (4) line integrals over closed paths are zero. Call ϕ the potential function. In a connected domain, the potential ϕ can be determined up to a constant: $\phi(x) = \int f \cdot d\alpha$ for a path α connecting a reference point x_0 to an arbitrary point in the connected domain.

Example: Newtonian Potential 3 dimensional. Consider a mass M at the origin, and another mass m at coordinates (x, y, z). Then the force field is defined on the domain $D = \mathbb{R}^3\backslash\{0\}$, $f(x, y, z) = -\frac{GMm}{r^3}\vec{r}$, $r = \sqrt{x^2 + y^2 + z^2}$. Now we can verify that f is conservative, and has a potential function given by $\phi(x, y, z) = \frac{GMm}{r}$.

Example: Arbitrary Conservative Field Consider a function $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, D open and connected. Consider a particle of mass m moving along a path $\alpha(t), t \in [0,1]$, $\alpha \in C^2$, $\alpha([0,1]) \subseteq D$. Let the particle move under the force f. Define the kinetic energy at time t to be $E(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m||\alpha'(t)||$, where $v(t)$ is the speed. Also, by Newton's second law, $f(\alpha(t)) = m\alpha''(t)$. THen the line integral $\int f \cdot d\alpha = \int f(\alpha(t)) \cdot \alpha'(t) = \int m\alpha''(t) \cdot \alpha'(t) = \int \frac{1}{2}m\frac{d}{dt}(\alpha'(t) \cdot \alpha'(t)) = \int \frac{1}{2}m\frac{d}{dt}(v(t))^2 = \frac{1}{2}m(v(1)^2 - v(0)^2) = E(1) - E(0)$. Note that if f is conservative, then $f = \nabla\phi$, then $\int f \cdot d\alpha = \phi(b) - \phi(a)$. We call (in physics) $-\phi$ to be the potential energy, sch that the sum of the potential and kinetic energy is a constant, since the integral equals $\phi(1) - \phi(0) = E(1) - E(0) \implies E(1) - \phi(1) = E(0) - \phi(0)$.

Necessary conditions for a vector field to be a gradient Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, D open. Let $f = (f_1, \ldots, f_n)$. If f is conservative, then $\forall 1 \leq i, j \leq n$, $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$. For example, in $n = 2$, we write $f = (P, Q)$, so we require $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Proof If $f = \nabla\phi$, then $f_i = \frac{\partial\phi}{\partial x_i}$. Since $\phi \in C^2$ (since f has continuous first order partial derivatives, which are the second order partial derivatives of ϕ), the mixed partials are equal, hence $\frac{\partial^2\phi}{\partial x_i\partial x_j} = \frac{\partial f_j}{\partial x_i} = \frac{\partial^2\phi}{\partial x_j\partial x_i} = \frac{\partial f_i}{\partial x_j}$.

Simply connected open set Suppose we have an open $U \subseteq \mathbb{R}^n$ which is connected. Then we say that U is simply connected if any continuous closed path contained in U can be continuously transformed inside U to a single point. This can be understood as U having no holes.

Example of a non-simply connected open set Consider $U = \mathbb{R}^2\backslash\{0\}$. U is not simply connected, since any circle centered at the origin cannot be shrunk to a point at the origin, since the origin is not contained in U.

Example of a typical simply connected open set: Convex Set A set U is convex if for any $a, b \in U$, the line connecting a to b is completely contained in U. A convex set is simply connected. W ecan pick any closed path α contained completely in U. We can shrink the curve using $\phi_s(t) = \alpha(t) - s[\alpha(t) - \alpha(0)], s \in [0,1]$ to shrink the curve to the point $\alpha(0)$.

Necessary and sufficient conditions for vector field to be a gradient Given $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, D open, $f \in C^2$. If D is simply connected, then the following are equivalent: (1) f is conservative (2) $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, 1 \leq i, j \leq n$.

Importance of being simply connected Consider the domain $D = \mathbb{R}^2 \backslash \{0\}$. D is not simply connected. Also let $f(x, y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}) = (P, Q)$. We note that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. However, f is not conservative. Note that we can calculate the line integral of f on the circle $\alpha(t) = (\cos t, \sin t), t \in [0, 2\pi]$. $\int f \cdot d\alpha = 2\pi \neq 0$. Hence although α was closed, the line integral along α was not zero. Hence f is not conservative.

Example Consider the vector field $f(x, y, z) = (z^3 + 2xy, x^2, 3xz^2)$ defined on $\mathbb{R}^3$. Calculate $\int f \cdot d\alpha$ on an ellipse $\alpha(t) = (a \cos t, b \sin t, 0)$ lying in the x-y plane.

Jordan Curve A Jordan curve, or a simple closed curve, is a 1-1 closed curve. A typical example is the circle/ellipse.

Jordan Curve Theorem If α is a Jordan curve in $\mathbb{R}^2$, and C is the graph of α, then $\mathbb{R}^2 \backslash C = U \cup V$ with U, V disjoint and open, with U bounded, V unbounded, and $C = \partial(U) = \partial(V)$. Call U the interior of C and V the exterior of C. Note that this differs from the interior/exterior in the context of open sets. The interior of a curve (which has content zero) is empty. But in the Jordan curve theorem, the interior is the set of points it encompasses. We call a Jordan curve positively oriented if transversing it, the interior is on the left. An alternative way to verify whether a domain is simply connected is to check that for every closed Jordan curve in the set, the interior of the curve is contained in the domain.

Chapter 7

Week 7

7.1 12 May 2014 Lecture

Green's Theorem Consider a Jordan curve $\alpha : [a, b] \to \mathbb{R}^2$, positively oriented, piecewise C^1. Call the interior of the curve U, call the curve C. Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^!$ vector field where $D \subseteq \mathbb{R}^2$ is open and contains the interior of the curve U and the curve C: $U \cup C \subseteq D$. Let $f = (f_1, f_2)$. Then $\int f \cdot d\alpha = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy$, where $R = U \cup C$. Note that we can substitute U for R, since the curve C has content zero, hence the integral on C is zero. Write $\oint f \cdot d\alpha$, with a small counterclockwise arrow on the circle for this integral. Alternative notation: $f = (P, Q), \alpha(t) = (x(t), y(t))$, then $\oint f \cdot d\alpha = \int_a^b (P, Q) \cdot (x'(t), y'(t))dt = \int_a^b P(x(t), y(t))x'(t)dt + \int_a^b Q(x(t), y(t))y'(t)dt$. Write $x'(t)dt = dx$ and $y'(t)dt = dy$. Hence this can be written as $int_a^b Pdx + Qdy$. Note that this does not depend on α, but only on C.

Other domains Consider the annulus (donut in 2D). Visualize a horizontal cut through the center. Then the top of the annulus above the cut obeys Green's theorem, as well as the bottom half. Hence apply Green's theorem separately to each piece. Call the upper half R_1 and the lower half R_2. Then $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. We note that the path traced along the cut will cancel out, since both halves traverse it in opposite directions. Let the inner curve be C_2 and the outer curve be C_1. Then the integral is $\oint_{C_1} Pdx + Qdy - \oint_{C_2} Pdx + Qdy$. Note we can do the same for two or more holes inside the curve. Just perform 2 or more cuts, with one cut across each hole.

Application of Green's Theorem: Calculation of Area Let C be a piecewise C^1 Jordan curve, given by a positive parametrization $\alpha : [a, b] \to \mathbb{R}^2$. Let the interior of the curve be U. By defintion, the area of the U is $A(U) = \iint_U dxdy$. Hence we want to obtain a vector field with $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Hence we just choose $f(x, y) = (0, x)$. Then $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = 0$, so their sum is 1. Hence we can apply Green's Theorem to obtain that $\iint_U 1dxdy = \oint_C Pdx + Qdy = \oint_C Qdy = \int_a^b Q(x(t), y(t))y'(t)dt = \int_a^b x(t)y'(t)dt$. Similarly, using the vector field $f(x, y) = (-y, 0)$, we have that $A(U) = -\oint ydx = -\int_a^b y(t)x'(t)dt$. We can also add the two formulae and divide by two to obtain $A(U) = \frac{1}{2}\int_a^b [x(t)y'(t) - x'(t)y(t)]dt$. Note that we can use this to determine if α is positively oriented or not. If α is positively oriented, then the sign of $\int_a^b x(t)y'(t)dt$ should be positive. If the sign of the integral is negative, then we have just chosen the wrong direction.

Proof of Green's Theorem for the simple rectangle case Note that by taking $Q = 0$ and then $P = 0$ respectively in Green's Theorem, we have that $-\oint_C Pdx = \iint_R \frac{\partial P}{\partial y} dxdy$ and $\oint Qdy = \iint_R \frac{\partial Q}{\partial x} dxdy$. Hence we can obtain Green's Theorem by adding up these two formula. It will suffice to prove the first formula, for the second one follows similarly. Consider a rectangle $[a, b] \times [c, d]$. Call the bottom line C_1, the right side C_2, the top line C_3 and the left line C_4. For C_1, we use the parametrization $C_1 : t \mapsto (a+t(b-a), c), t \in [0, 1]$, $C_2 : t \mapsto (b, c+(t-1)(d-c)), t \in [1, 2]$, $C_3 : t \mapsto (b+(t-2)(a-b), a), t \in [2, 3]$, $C_4 : t \mapsto (a, d + (t - 3)(c - d)), t \in [3, 4]$. Call these paths $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ respectively. Then $\oint_C$ is the sum of the integrals over each individual curve (line). Now we have chosen that $Q = 0$, hence the integral $\oint Pdx = \int_{C_1} Pdx + \int_{C_3} Pdx$, since the integral along C_2 and C_4 faces a constant value (since the value of x is fixed on each line), hence integrates to zero (?). Consider the equivalent parametrization for α_1, $\beta_1 : t \mapsto (t, c), t \in [a, b]$ and for $-\alpha_3$, $\beta_2 : t \mapsto (t, d), t \in [a, b]$. Then $\oint_C Pdx = \int_a^b P(t, c)dt - \int_a^b P(t, d)dt$. Hence $-\oint_C Pdx = \int_a^b (-P(t, c) + P(t, d))dt$. Now we calculate $\iint_R \frac{\partial P}{\partial y} dxdy$ using Fubini's theorem. Then this is $\int_a^b \left(\int_c^d \frac{\partial P}{\partial y}(x, y)dy \right) dx$. By the fundamental theorem of calculus in one variable, this is equal to $\int_a^b [P(x, d) - P(x, c)]dx$. Hence, comparing this result with the one for $-\oint_C Pdx$, we obtain that $-\oint_C Pdx = \iint_R \frac{\partial P}{\partial y} dxdy$. Note that this proof can be used for any curve made of lines parallel to the x or y-axes.

Theorem Consider $D \subseteq \mathbb{R}^2$ which is simply connected and open. Then let $f = (P, Q) : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 Then f is conservative iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D.

Proof Now we have proven the forward direction last week. Now we prove the reverse direction. For any Jordan curve α in D that satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we have that $\oint_C f \cdot d\alpha = 0$. The rest of this proof is showing that this works for any curve.

Winding numbers Consider α as a piecewise C^1 Jordan curve, and consider a point $z_0 = (x_0, y_0) \notin \alpha([a, b])$. Then the winding number of C relative to z_0 is $W(\alpha, z_0) = W(C, z_0) = \frac{1}{2\pi} \int_C \left[-\frac{y-y_0}{r^2} dx + \frac{x-x_0}{r^2} dy\right]$, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. If α is just a normal curve (not necessarily Jordan) W turns out to be an integer, and tells you how many times α goes around z_0, and the sign tells you if it goes around in the counterclockwise direction or clockwise. For a Jordan curve, W is either $-1, 1$ or 0. W is $+1$ if z_0 is in the interior and α is positively oriented, -1 if z_0 is in the interior and α is negatively oriented, and 0 if z_0 is in the exterior of C.

7.2 Lecture 14 May 2014

Del operator Consider $\phi : D \subseteq \mathbb{R}^n \to \mathbb{R}$, D open. Then ϕ has particle derivatives $\nabla\phi = (D_1\phi, \ldots, D_n\phi)$ which is a vector field: $\nabla\phi : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. We define the formal vector $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Divergence Consider $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, with D open, and such that the partials of f exist. Define $div(f) = \nabla \cdot f$. We note that this is equal to the trace of the Jacobian.

Laplacian Define $\Delta = \nabla \cdot \nabla$. We require that ϕ has second derivatives that exist in order for the Laplacian to be applied to it. We note that the Laplacian is the trace of the Hessian. Now the Laplacian can also be applied to a vector field. Consider $F = (F_1, \ldots, F_n)$. Then $\Delta F = (\Delta F_1, \ldots, \Delta F_n)$.

Harmonic We define ϕ to be harmonic if it has derivatives of second order and $\Delta\phi = 0$, which is the Laplace equation.

Example Let $\phi(x_1, \ldots, x_n) = r^\alpha$, where $\alpha \in \mathbb{R}$ and $r = ||(x_1, \ldots, x_n)||$ for $x \in \mathbb{R}^n \backslash \{0\}$. For example, in the Newtonian potential, we have that $n = 3$ and $\alpha = -1$. We note that $\frac{\partial\phi}{\partial x_i} = \alpha r^{\alpha-1} \frac{\partial r}{\partial x_i} = \alpha r^{\alpha-1} \frac{x_i}{r} = \alpha r^{\alpha-2} x_i$. Also, the second derivative is $\alpha(\alpha-2)r^{\alpha-4}x_i^2 + \alpha r^{\alpha-2}$. Hence $\Delta\phi = \alpha(\alpha-2)r^{\alpha-2} + n\alpha r^{\alpha-2}$. Now $\Delta\phi = 0$ iff $\alpha(\alpha-2) + n\alpha = 0$. This requires $\alpha = 0$ or $n = 2 - \alpha$. Hence when $n = 3$, we require $\alpha = -1$ for the potential to be a harmonic function.

Cross Product (in 3D) Suppose we have $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Then $u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$. Also note that $u \times v = -v \times u$.

Curl This only makes sense when $n = 3$. Consider $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, D open, with F having partial derivatives. Write $F = (P, Q, R)$. Then $curl(F) = \nabla \times F$. The curl is a vector field $\nabla \times F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. If $\nabla \times F = 0$, then F is called irrotational.

Proposition Suppose $\phi : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a C^2 scalar field, and $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 vector field, D open. Then (1) $\nabla \times (\nabla\phi) = 0$ (the curl of a gradient is zero) (2) $\nabla \cdot (\nabla \times F) = 0$ (divergence of a curl is zero).

Proof Expand out $\nabla \times (\nabla\phi)$, and note that the mixed derivatives are equal, since $\phi \in C^2$. Same thing for the second proposition.

Theorem Given $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, D open and simply connected, $F \in C^2$. Then TFAE: (1) F is conservative (i.e. $F = \nabla\phi$ for some ϕ. (2)$\nabla \times F = 0$ (i.e. F is irrotational.

Proof (2) $\implies$ (1): Write out the components of the curl, set it to zero. This is exactly the condition for F to be conservative in a simply connected set: $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$.

Example Consider $D = \mathbb{R}^3 \backslash \{0\}$. Note that D is simply connected, since you can shrink any curve to a single point, unlike the 2D case. Let $G(x, y, z) = \frac{xi+yj+zk}{r^m}, m \geq 1$. Then $\nabla \cdot G = \frac{3-m}{r^m}$. If $m \neq 3$, then $\nabla \cdot G \neq 0$. But if $m = 3, \nabla \cdot G = 0$.

Theorem Suppose $D \subseteq \mathbb{R}^3$ is an open box and an open ball on all of $\mathbb{R}^3$ (needs to be a two-simply connected set, which is a set such that for any closed surface, its interior is also contained in the set. Then for any vector field $G : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ in C^1, TFAE: (1) $\nabla \cdot G = 0$, (2) $\exists F : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ in C^2 such that $\nabla \times F = G$. We call F a vector potential for G.

Solenoidal If $\nabla \cdot G = 0$, we call G solenoidal.

7.3 Lecture 16 May 2014

Example: Finding F such that its curl is G Consider a $G : \mathbb{R}^3 \to \mathbb{R}^3$. If $\nabla \cdot G = 0$, by Theorem above, there is some F such that $\nabla \times F = G$. Now we can choose $F = (P, Q, R)$ such that $R = 0$. We can do this because $\nabla \times (F + \nabla\phi) = \nabla \times F$ since the curl of a gradient is zero, and we can choose a gradient with the third coordinate being $-R$. Then we can solve for $F + \nabla\phi$ with its third coordinate zero.

Change of variables for one variable Consider an $\phi : [a, b] \to [c, d]$ which is a change of variables, ϕ is one-one and onto, with non-zero and continuous derivatives. Then ϕ' is either strictly positive (increasing function) or strictly negative (decreasing function). Hence we can make the change of variables $x = \phi(y), dx = \phi'(y)dy$, so $\int_c^d f(x)dx = \int_{\phi^{-1}(c)}^{\phi^{-1}(d)} f(\phi(y))\phi'(y)dy$. Hence this is equal to $\int_a^b f(\phi(y))\phi'(y)dy$ if $\phi' > 0$ and $\int_a^b f(\phi(y)(-\phi'(y))dy$ if $\phi' < 0$. In both cases, the integral is $\int_a^b f(\phi(y))|\phi'(y)|dy$.

Diffeomorphism Consider U, V open sets in $\mathbb{R}^n$, with $\partial(U)$ and $\partial(V)$ have content zero. Then a one-one correspondence $\phi : U \to V$ such that both the function and its inverse function are C^2 is called a diffeomorphism. Note that the determinant of the Jacobian of a diffeomorphism is non-zero (recall that $\phi \circ \phi^{-1} = I$, and by the chain rule $D(\phi) \cdot D(\phi^{-1}) = I$. Taking the determinant of both sides, we have that $det(D\phi)det(D(\phi^{-1})) = 1$. Hence the determinant is non-zero) Write this as $det(D\phi(u)) \neq 0, \forall u \in U$. Alternative interpretation: ϕ is smooth and invertible with ϕ^{-1} smooth as well.

Inverse function theorem If $\phi : U \to V$ is a C^1 function, bijective and $\det D(\phi(u)) \neq 0$, then ϕ is a diffeomorphism.

Notation Let $\phi = (\phi_1, \dots, \phi_n)$. Then the Jacobian determinant of ϕ is denoted by $\frac{\partial(\phi_1,\dots,\phi_n)}{\partial(u_1,\dots,u_n)}$ when defined on a bounded open set $U = (u_1, \dots, u_n)$.

Change of variables formula Consider U, V open sets with boundaries of content zero. Consider a diffeomorphism $\phi : U \to V$ (1-1, onto, C^1, $detD(\phi(u)) \neq 0, \forall u \in U$) and $detD(\phi(u))$ bounded. Then for a continuous bounded function f, $\int_V f(x_1, \dots, x_n)dx_1 \cdots dx_n = \int_U f(\phi(u))|detD\phi(u)|du_1 \cdots du_n$.

Intuitive proof Since U and V have boundary of content zero, we can replace them with their closures, since the boundary would not contribute to the integral. We can also write $x_i = \phi_i(u_1, \dots, u_n)$ for $(u_1, \dots, u_n) \in U$ and $(x_i, \dots, x_n) \in V$. Then $dx = dx_1 \cdots dx_n = |detD\phi(u)|du_1 \cdots du_n = |detD\phi(u)|du$. Also, $Vol(V) = \int_U |detD\phi(u)|du_1 \cdots du_n$.

Examples Consider $\phi = T$, a linear transformation. Then $D\phi = m(T), det(D\phi) = det(m(T)) \neq 0$, hence T is invertible. Then $dx = |det(m(T))|du$. Also $Vol(V) = |det(m(T))|Vol(U)$.

Polar Coordinates Given a point (x, y) define $r = \sqrt{x^2 + y^2}$ and θ such that $x = r\cos\theta, y = r\sin\theta$. We choose $r \geq 0$ and $\theta \in [0, 2\pi]$. Consider the change of coordinates function $\phi(r, \theta) = (x, y)$. Then $\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta$, and so on. The Jacobian determinant is r. Hence we replace $dxdy$ with $rdrd\theta$. Then the integration formula is $\iint_V f(x, y)dxdy = \iint_U f(r\cos\theta, r\sin\theta)rdrd\theta$.

Chapter 8

Week 8

8.1 19 May 2014 Lecture

Example: Cylindrical coordinates Consider $(r, \theta, z) \mapsto (x, y, z)$ under the change of variables ϕ. $x = r\cos\theta, y = r\sin\theta, z = z$. Then the Jacobian determinant is r. Hence when integrating, we replace $dxdydz$ with $rdrd\theta dz$.

Example: Spherical coordinates When integrating we replace $dxdydz = r^2 \sin\theta drd\theta d\phi$.

Parametrized surfaces Consider only surfaces in $\mathbb{R}^3$. A parametrized surface consists of a set $D \subseteq \mathbb{R}^2$ of the form $D = U \cup C$, where C is a piecewise C^1 Jordan curve and U is the interior of C and a continuous map $\Phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ such that $\Phi(u, v) = (X(u, v), Y(u, v), Z(u, v))$. Hence the image of Φ on D is a surface in $\mathbb{R}^3$. If Φ is one-to-one on U, we call the surface simple. Call the surface parametrized by Φ as S. Also, if Φ is one-to-one on the Jordan curve C, then the image of C under Φ is also a Jordan curve in $\mathbb{R}^3$.

Example: Explicit Parametrization Consider $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ as a C^1 function. Then the surface S is the graph of f on D. Then the parametrization is $\Phi(u, v) = (u, v, f(u, v))$. This parametrization is clearly one-one.

8.2 Lecture 21 May 2014

Another parametrization of a hemisphere (Stereographic Projection) Consider the southern hemisphere of the unit sphere. Take the point with coordinate $(0, 0, 1)$, and connect this point with an arbitrary point on the hemisphere, and identify the point (u, v) where this line intersects the xy plane. This is a one-to-one correspondence between the points on the unit disk and the points on the hemisphere. We can write the formula for this mapping $\Phi(u, v) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2 + v^2 - 1)$.

Plane parametrization Consider the plane containing the point (x_0, y_0, z_0) with normal vector $(a, b, c) \neq 0$. Then the equation of the plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. Take two vectors that are not lying on the same line (linearly independent). Then we can use the following parametrization $\Phi(u, v) = (x_0, y_0, z_0) + up + vq$, $(u, v) \in \mathbb{R}^2$.

Surface given by a level set Consider the function $f(x, y, z) = 0$, and a point where the gradient is non-zero. Then we can represent the surface locally using the implicit function theorem.

Definition: Regular Point and Regular Parametrization Given $(u_0, v_0) \in U$, we say that (u_0, v_0) is a regular point for Φ if $\frac{\partial\Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0, v_0)$ are independent. That is the same thing as saying that the cross-product is not the zero vector. Note that the cross product of these vectors is a normal to the surface a the point $\Phi(u, v)$. Call the normal vector $n(u_0, v_0)$. Hence we can define the tangent plane at that point to be $n(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$, where $x_0 = X(u_0, v_0), y_0 = Y(u_0, v_0), z_0 = Z(u_0, v_0)$. If every point in U is regular we call Φ regular or smooth.

Example: Explicit Parametrization Given $\Phi(u, v) = (u, v, f(u, v))$, we have that $\frac{\partial\Phi}{\partial u} = (1, 0, \frac{\partial f}{\partial u})$, and $\frac{\partial\Phi}{\partial u} = (0, 1, \frac{\partial f}{\partial v})$. The cross product of these vectoris is $(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$ which is not zero, since the z-coordinate is always 1. Hence when you have an explicit parametrization, then the cross product of the derivative is never zero, hence all points are regular points, and the parametrization is regular. Also, the normal always points in the positive z-direction, since the z-coordinate is 1. Note that we can think of the surface as the level set of the function $h(u, v, z) = z - f(u, v)$ corresponding to $h = 0$. Then the normal to the level surface is the gradient of $h(u, v, z)$, which is $\nabla h = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$. We note that this is the same vector as the one obtained from the cross product. Hence this concept of the normal is consistent.

Area of a parametrized curve Take Φ to be a simple (one-to-one) regular parametrized surface. Then the tangent plane at some point (u_0, v_0) is a reasonable approximation of the surface around that point. If we increase u by Δu, then $\Phi(u_0 + \Delta u, v_0) \approx \Phi(u_0, v_0) + \frac{\partial \Phi}{\partial u}(u_0, v_0)\Delta u$. Similar equation for $\Phi(u_0, v_0 + \Delta v)$. Then the rectangle with dimension $\Delta u, \Delta v$ in uv-space becomes a parallelogram defined by the vectors $\frac{\partial \Phi}{\partial u}\Delta u, \frac{\partial \Phi}{\partial v}\Delta v$ with area $\Delta S = \left|\left|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right|\right| \Delta u \Delta v$. We define the area of the parametrized surface to be the integral $a(\Phi) = \iint_U \left|\left|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right|\right| dudv$.

Surface Integrals over Scalar Fields Given D, Φ as before, and $f : R \subseteq \mathbb{R} \to \mathbb{R}^3$ a bounded function defined on some subset of $\mathbb{R}^3$, with $\Phi(D) \subseteq R$. Then $\iint_\Phi f dS = \iint_U f(\Phi(u, v)) \left|\left|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right|\right| dudv$.

8.3 Lecture 23 May 2014

Surface Integral over Vector Fields (Flux of a vector field) Consider a vector field $F : P \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, surface $S \subseteq P$ and a regular parametrization Φ at (u_0, v_0). Then the normal to the surface at (u_0, v_0) is given by the cross product $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}$, which is non-zero (by defintion of the regular point. Hence we can consider the unit vector in the normal direction. Write this as $n(u_0, v_0) = \frac{\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}}{||\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}||}$. We define the integral (or flux) of F over Φ to be $\iint_\Phi (F \cdot n) dS = \iint_U F(\Phi(u, v)) \cdot n(u, v) ||\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}|| dudv = \iint_U F(\Phi(u, v)) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) dudv$.

Invariance under change of parameters Consider the domain D in uv-space with a parametrization Φ that maps this domain onto $\mathbb{R}^3$, forming the surface S. Now consider a diffeomorphism ϕ from domain U in st-space to range V in uv-space. Then $det(D\phi) \neq 0$ since ϕ is a diffeomorphism. Then we have a parametrization of S using $\Psi(s, t) = \Phi(\phi(s, t)), \Psi = \Phi \circ \phi$. Then $\iint_\Phi f dS = \iint_\Psi f dS$. The flux is invariant under change of parameters provided $\det D(\phi) > 0$, which means the change is orientation preserving.

Sketch of proof By the chain rule, $\frac{\partial \Psi}{\partial s} \times \frac{\partial \Psi}{\partial t} = \frac{\partial \Phi}{\partial u}(\phi(s, t)) \times \frac{\partial \Phi}{\partial v}(\phi(s, t)) \det(D\phi(s, t))$.

Pappus' Theorem for Surfaces Let C be the graph of $z = f(x) > 0,\ f \in C^1, f : [a, b] \to \mathbb{R}^2$. Now we rotate this graph around the z-axis, obtaining a surface. Call S the surface of revolution of C around the z-axis. Then Pappus' theorem says that the area of S is $a(S) = 2\pi Lh$, where L is the length of C, and h is the distance of the centroid of C to the z-axis. Note that we can parametrize S using polar coordinates of the projection $\Phi(u, v) = (u \cos v, u \sin v, f(u))$. Note that $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = (-uf'(u) \cos v, uf'(u) \sin v, u)$. Hence the norm is $u\sqrt{1 + f'(u)^2}$. Hence $a(S) = \int_0^{2\pi} \int_a^b u\sqrt{1 + f'(u)^2} dudv = 2\pi \int_a^b x\sqrt{1 + f'(x)^2} dx$. Then we can parametrize the curve C using $\alpha(x) = (x, f(x)), a \leq x \leq b$. So $ds = \sqrt{1 + f'(x)} dx$, so $a(S) = 2\pi \int_\alpha x ds = 2\pi \bar{x} L$ because by definition of $\bar{x} = \frac{\int_\alpha x ds}{L}$.

Chapter 9

Week 9

9.1 28 May 2014 Lecture

Stokes' Theorem Consider a positively oriented piecewise C^1 Jordan curve γ, $\gamma : [a, b] \to \mathbb{R}^2$ and a parmetrization Φ such that the image of Φ on $D = U \cup C$, where U is the interior of γ and C is the graph of γ, forms the surface S. Let Φ be regular on U and also be C^2 on some open set containing D. Let $\alpha = \Phi \cdot \gamma$ be the path traced out by γ under the parametrization Φ. Let F be a C^1 vector field defined on some open set R, with $S \subseteq R$. $F : R \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. Then $\oint_\alpha F \cdot d\alpha = \iint_\Phi (\nabla \times F) \cdot ndS$. The integral $\oint_\alpha F \cdot d\alpha$ is called the circulation of F on α.

Relation between Green's Theorem and Stokes' Theorem Take Φ to be the "identity" parametrization $\Phi(u, v) = (u, v, 0)$. Let $S = D$ and let $F : R \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. Let $\tilde{F}(u, v, w) = (F(u, v), 0)$ be the three-dimensional analogue of F. Let $F = (P, Q)$. Then $\tilde{F} = (P, Q, 0)$. Then Stokes' Theorem says that $\iint_\Phi (\nabla \times \tilde{F}) \cdot ndS = \oint \tilde{F} \cdot d\gamma$, where $\gamma(t) = (\alpha(t), 0)$. Now the third coordinate of $\nabla \times \tilde{F}$ is $\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}$. We note that the unit normal vector n is in the z-direction, so Stokes' theorem becomes $\iint_U \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) dudv = \oint_\alpha (P, Q) \cdot d\alpha$, which is just Green's Theorem.

Curl zero function If F is C^1 and $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\nabla \times F = 0$ everywhere, then $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all $0 \leq i, j \leq 3$ since F is conservative. This shows that $\oint F \cdot d\alpha = 0$ for any closed curve α.

Example The integral of a curl on a closed surface is zero: $\iint (\nabla \times F) \cdot ndS = 0$.

9.2 29 May 2014 Recitation

Homeomorphism A function is continuous, the inverse exists, and the inverse is continuous.

Diffeomorphism A function is infinitely smooth, the inverse exists, and the inverse is also infinitely smooth.

9.3 30 May 2014 Lecture

Example Consider the upper unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$. The boundary of the hemisphere is the unit circle on the xy-plane, call it C. We can parametrize the boundary using $\gamma(t) = (\cos t, \sin t)$. We can also parametrize the upper hemisphere using the stereographic projection. Pick the point $(0, 0, -1)$, and draw a line connecting $(0, 0, -1)$ to a point on the unit disk on the xy plane, then extrapolate it to hit the hemisphere. The domain of the parametrization is the unit disk. Then the corresponding $\alpha(t) = (\cos t, \sin t, 0)$. Consider a vector field defined on all of $\mathbb{R}^3$ as $F(x, y, z) = (\cos x \sin z, x^3 + xy, e^{x+y} + e^{y^2+z^2})$. We want to calculate the surface integral $\iint_\Phi (\nabla \times F) \cdot ndS$ over the hemisphere. By Stokes' theorem, we know that the integral is equal to $\oint F \cdot d\alpha$. Note too that by Stokes' theorem $\oint F \cdot d\alpha$ is also the surface integral of $\nabla \times F$ over any surface that shares the same boundary as the hemisphere. We choose the surface to be the unit disk, with the "identity" parametrization $(u, v) \mapsto (u, v, 0)$ for $u^2 + v^2 \leq 1$. Hence in this case, the unit normal vector is just $(0, 0, 1)$. Hence we are only interested in the z-coordinate of the curl, which is $3x^2 - x$. Hence the integral is just $\iint_{U=\text{unit disk}} (3x^2 - x) dxdy$. Using polar coordinates $x = r \cos \theta, y = r \sin \theta, r \in (0, 1], \theta \in [0, 2\pi)$, the integral becomes $3\pi/4$.

Gauss' Theorem/Divergence Theorem Consider a bounded region $V \subseteq \mathbb{R}^3$ that is enclosed by a boundary S which can be decomposed into finitely many pieces $S = S_1 \cup \cdots \cup S_m$ where S_i is a parametrized surface and S_i, S_j do not overlap except at their boundaries. Then for any C^1 vector field F which is defined on some open set containing $V \cup S$, we have that $\iint (F \cdot n) dS = \sum_{i=1}^{m} \iint_{S_i} (F \cdot n) dS = \iiint_V (\nabla \cdot F) dxdydz$ provided that the normal vector to S at every point on S points outside V.

Example If $F = \nabla \times G$ is solenoidal, then $\nabla \cdot F = \nabla \cdot (\nabla \times G) = 0$, so $\iint_S (\nabla \times G) \cdot n dS = \iint_S (F \cdot n) dS = 0$.

Two concentric spheres Consider spherical surfaces S_1 and S_2 with S_1 contained inside S_2, and consider V to be the solid between S_1 and S_2. Then we can write $\iiint_V (\nabla \cdot F) dxdydz = \iint_{S_2} (F \cdot n) dS - \iint_{S_1} (F \cdot n) dS$ since the direction of the unit outward normal is in the opposite direction for S_2 and S_1. The unit normal to the surface is outwards away from the origin at S_2 and inwards towards the origin at S_1. Note that in the equation $\iint_{S_2} (F \cdot n) dS - \iint_{S_1} (F \cdot n) dS$, the normal n is pointing outwards away from the origin, since the minus sign is always included. Note that if $\nabla \cdot F = 0$ in the region V, then $\iint F \cdot n dS = 0$, and $\iint_{S_2} (F \cdot n) dS = \iint_{S_1} (F \cdot n) dS$, so the surface integrals through the individual surfaces are the same.

Chapter 10

Week 10

10.1 Lecture 2 June 2014

Example Consider S to be the unit sphere, V the unit ball, and compute $\iint_S (x^2+y+z)dS$. Let n be the unit normal to S pointing outwards, which is (x,y,z). We need to find a vector field F such that $F \cdot n = x^2+y+z$. We write $F=(F_1,F_2,F_3)$, so $F=(x,1,1)$. Hence $\iint_S (x^2+y+z)dS = \iint_S (F\cdot n)dS = \iiint_V \nabla\cdot(x,1,1)dxdydz$ by the divergence theorem. The result is just the volume of the unit ball, $4\pi/3$.

Solving for curl of zero divergence vector field Let F be C^1 on $\mathbb{R}^3$. If $\nabla\cdot F=0$, then $F=\nabla\times G$ for some vector function G. Suppose $F=(F_1,F_2,F_3)$. Then $G_1=\int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,y,0)dt, G_2 = -\int_0^z F_1(x,y,t)dt, G_3=0$ plus any gradient $\nabla f(x,y,z)$ because the curl of a gradient is zero.

10.2 Lecture 4 June 2014

Implicit function theorem: Motivation Let C be a curve in $\mathbb{R}^2$. Often, C is given as the graph of a function (which can be a function of x or y). But there are some natural curves that are not the grpah of a function, such as the circle. More generally, consider a curve C which is given by $\{(x,y):\phi(x,y)=0\}$, the level set of some function ϕ. Roughly, the implicit function theorem says that if ϕ is "nice", then locally (which means that there is a neighborhood) C looks like the graph of a C^1 function. Intuitively, if $\phi\in C^2$, then locally there is a tangent plane T at each point. As long as T is not vertical (i.e. $\frac{\partial\phi}{\partial y}(x_0,y_0)=0$), then we can use it to write C as the graph of a function near (x_0,y_0).

Statement of IFT in two-dimensions Let $D\subseteq\mathbb{R}^2$ be open and let $\phi: D\to\mathbb{R}$ be C^1. Suppose that there exists a point $(x_0,y_0)\in D$ such that $\phi(x_0,y_0)=0$. This is just the point on the surface we are starting from. Also suppose that $\frac{\partial\phi}{\partial y}(x_0,y_0)\neq 0$. Then there exists open U with $x_0\in U$ and V with $y_0\in V$ such that $\exists! f: U\to V$, $f\in C^1$ such that $\phi(x,y)=0 \iff y=f(x)$. This is also applicable when we exchange the roles of x and y.

Corollary: The Derivatives of f $\forall x\in U, \phi(x,f(x))=0$. By the chain rule, we differentiate both sides with respect to x to obtain $\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}f'(x)=0$, with the derivatives of ϕ evaluated at $(x,f(x))$. So $f'(x)=\frac{-\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}(x,f(x))$. Hence we can use this to know the linearization of $f(x)=f(x_0)+(x-x_0)f'(x_0)+o(|x-x_0|)$, since $f(x_0)=y_0$ and $f'(x_0)$ is calculated using this Corollary.

Corollary II If C is a compact curve such that $\nabla\phi\neq 0$ on C, then we can write $C=\cup_{i=1}^n G_i$, where G_i is the graph of a C^1 function.

Statement of IFT in n-dimensions Let $D\subseteq\mathbb{R}^{n+m}$, and $F: D\to\mathbb{R}^m$ such that $(\underline{x},\underline{y})\mapsto(\phi_1(\underline{x},\underline{y}),\dots,\phi_m(\underline{x},\underline{y}))$ with $\underline{x}=(x_1,\dots,x_n)$ and $\underline{y}=(y_1,\dots,y_m)$. Suppose that there exists a $(\underline{x_0},\underline{y_0})\in D$ such that $F(\underline{x_0},\underline{y_0})=0$. Also assume that $\det\left(D_{\underline{y}}F(\underline{x_0},\underline{y_0})\right)=\det\left(\frac{\partial\phi_i}{\partial y_j}\right)\neq 0, 1\leq i,j\leq m$. Then there exists open U with $\underline{x_0}\in U$ and open V with $\underline{y_0}\in V$ and $\exists! f: U\to V$ with $f\in C^1$ such that $F(\underline{x},\underline{y})=0 \iff \underline{y}=f(\underline{x})$. Notice the parallel to the two-dimensional case, since a real number is non-invertible if and only if it is zero. Hence we require that the Jacobian is invertible, so its determinant is non-zero.

Inverse function Theorem Let $D\subseteq\mathbb{R}^n$ be open. Let $G: D\to\mathbb{R}^n$ be C^1. Suppose that $\det DG(\underline{a})\neq 0$ for $a\in D$. Then there exists an open U that contains $\underline{a}$ and V that contains $b=G(\underline{a})$ such that $G: U\to V$ is bijective, and G^{-1} is C^1.

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Rotation Matrix For an anticlockwise rotation in θ, we have that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and the determinant is 1.

Rotation in the xy plane $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation in the xz plane $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$.

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Example: Conservative Forces and Path independence Consider a path $\alpha(t) := (t^2, t^3\cos(\pi t))$ for $t \in [0,1]$ which traces out the curve C. Let $F(x,y) = (y,x)$ be a vector field. Compute $\int F \cdot d\alpha = \int_0^1 F(\alpha(t)) \cdot \alpha'(t)dt$. Note that F is continuous since it is the composition of continuous functions. F is also defined on all of $\mathbb{R}^2$. Also, the domain $\mathbb{R}^2$ is simply connected. We note that $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1} = 1$. Hence F is conserative. Since F is conservative, the line integral of F is path-independent. We can hence choose a straight-line path connecting the start and end points of C. We note that the start point is $(0,0)$ and the end point is $(1,-1)$. We choose $\beta(t) = (0,0) + t(1,-1), t \in [0,1]$ that connects the two points using a straight line. Then $\beta'(t) = (1,-1)$. So the integral is $\int_0^1 F(\beta(t)) \cdot (1,-1)dt$.

Example 2 Let S be the unit sphere in $\mathbb{R}^3$. Compute $\iint_S (3x^2 - y^2 + z^2)dS$. We note that the surface S is a closed surface. Also, the function $F(x,y,z) = (3x, -y, z)$ is such that $F(x,y,z) \cdot n = (3x,-y,z) \cdot (x,y,z)$. So the integral is $\iint_S (3x,-y,z) \cdot n dS$. We note that F is C^1 since it is made up of polynomials. Then by the divergence theorem, the integral is $\iiint_V \nabla \cdot (3x,-y,z) dxdydz = \iiint_V (3) dxdydz = 4\pi$.

Alternative solution to Example 2 Note that $\iint_S x^2 dS + \iint_S y^2 dS = \iint_S z^2 dS$ by symmetry. Formally, we can apply a coordinate change that exchanges any two coordinates. This coordinate change has determinant 1. Hence the Jacobian determinant is 1, and the integral will be the same if we exchange between x, y, z. Hence the integral is just $\iint_S x^2 + y^2 + z^2 dS$ by linearity of the integral. But $x^2 + y^2 + z^2 = 1$ since this is the unit sphere. Easy.

Example 3: Another symmetry argument Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. We note that the square of the integral is $\int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$. Write this in polar coordinates. Then the integral is $\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \int_0^{2\pi} \left[-e^{-r^2}/2 \right]_0^{\infty} = \int_0^{2\pi} e^0/2 = \pi$. Hence the square root of this is $\sqrt{\pi}$.