Ma1b Notes

1.1 08 Jan 2014 Lecture

Definition Let A be a set and $*$ be a binary operation on A; that in $*$, $A \times A \to A$, and $a * b$ denotes the image of $(a, b) \in A \times A$ under $*$. Call $a * b$ the product of a and b. $*$ is associative if $\forall a, b, c \in A, a * (b * c) = (a * b) * c$. $*$ is commutative if $\forall a, b \in A, a * b = b * a$.

Definition Identity for $*$ is some $e \in A$ such that $\forall a \in A, a * e = a = e * a$. Note that the identity multiplies to the left and right to the same effect.

Lemma 1A A binary operation has at most one identity.

Proof of Lemma Let e, f be identities for $*$. Then $f = e * f = e$, hence $f = e$, and there is only one identity.

Definition The inverse is an element $b \in A$ for a such that $a * b = e = b * a$, if e is the identity for $*$.

Lemma 1B A function $f : X \to X$ has an inverse with respect to composition $\circ$ iff f is a one-one correspondence. A function f is a one-one correspondence if $\forall y \in X, \exists$ a unique $x \in X$ with $f(x) = y$.

Proof of Lemma $\Leftarrow$: Suppose f is a one-one correspondence. Define $g : X \to X$ mapping $f(x) \mapsto x$. We claim that g is an inverse for f. That is, $f \circ g = id_x = g \circ f$. Note that $(g \circ f)(x) = g(f(x)) = x = id_x(x)$, hence $g \circ f = id_x$. Same argument applies for $(f \circ g)$.

Lemma 1C If $*$ is an associated binary operation then each $a \in A$ has at most one inverse.

Proof of Lemma Suppose there are two inverses b, c for a. Then $a * b = e = c * a$ by the definition of the inverse. Also, $c = c * e = c * (a * b) = (c * a) * b = e * b = b$. Hence $c = b$, and the inverse is unique.

Definition Each vector space has an associated "coordinate field" (i.e. $\mathbb{R}$ or $\mathbb{C}$ in Ma1b, which we refer to as $\mathbb{F}$). The vector space also has an associated set called V. The members of V are called vectors, and the members of $\mathbb{F}$ are called scalars. There is also scalar multiplication of $\mathbb{F}$ on V, which is a function $\mathbb{F} \times V \to V$ with $(a, v) \mapsto a \cdot v$. A vector space (or linear space) over $\mathbb{F}$ consists of a set V together with binary operations $+$ called addition and scalar multiplication $\cdot$ of $\mathbb{F}$ on V such that the following axioms hold:

1. Commutativity: $\forall x, y \in V, x + y = y + x$
2. Associativity: $\forall x, y, z \in V, (x + y) + z = x + (y + x)$
3. Existence of Identity/Zero vector O
4. Existence of Inverse: $\forall v \in V, \exists - vs.t.v + (-v) = O$
5. $a \cdot (b \cdot v) = (ab) \cdot v$
6. $a(u + v) = au + av$
7. $(a + b)u = au + bu$
8. $1 \cdot v = v$

Note that Axiom 1 in Apostol (closure under addition) just says that addition is a binary operation in V, while Axiom 2 says the same for scalar multiplication.

1.2 09 Jan 2014 Recitation

Definition A field $\mathbb{F}$ is a set with two binary operations addition and multiplication such that it satisfies the following axioms:

1. Associativity of addition: $(a + b) + c = a + (b + c) \forall a, b, c \in \mathbb{F}$
2. Associativity of multiplication

3. Commutativity of addition

4. Commutativity of multiplication

5. Existence of additive identity $\exists O \in \mathbb{F}, s.t. a + O = a = O + a \forall a \in \mathbb{F}$

6. Existence of multiplicative identity $\exists 1 \in \mathbb{F}, s.t. 1 \cdot a = a = a \cdot 1 \forall a \in \mathbb{F}$

7. Existence of additive inverse $\forall a \in \mathbb{F}, \exists b \in \mathbb{F} s.t. a + b = 0$

8. Existence of multiplicative inverse $\forall a \in \mathbb{F}, \exists c \in \mathbb{F} s.t. a \cdot c = 1 = c \cdot a$

9. Distributivity $a \cdot (b + c) = a \cdot b + a \cdot c$

Examples of Fields: $\mathbb{F} = \{0, 1, 2\}$ with addition and multiplication mod 3. $\mathbb{F} = \{0, 1, \ldots, p - 1\}$ with addition and multiplication mod p, where p must be a prime.

Checking if a vector space U is a subspace of V:

1. U is non-empty

2. If $u, v \in U$, then $u + v \in U$

3. If $a \in \mathbb{F}, u \in U$, then $a \cdot u \in U$

2 and 3 are equivalent to $au + bv \in U, \forall a, b \in \mathbb{F}, u, v \in U$.

Any vector space V has at least 2 subspaces, namely: $\{0\}$ and V.

1.3 10 Jan 2014 Lecture

Function spaces Let X be a set and definte F^X to be the set of all functions from $X \to F$. Define the addition and multiplication to be for $f, g \in F^X, x \in X, c \in F$, then $(f + g)(x) = f(x) + g(x)$, where the addition is in F. $(c \cdot f)(x) = c \cdot f(x)$. F^X is a function space. The zero vector in F^X is the zero-function $O : X \to F, x \mapsto 0, x \in X$. Subspaces of this function space (with restriction that X is a closed interval on $\mathbb{R}$): Space of all continuous functions on X, space of all differentiable (or integrable) functions on X, m by n matrices over F.

Linear Span Let $S \subseteq V$. The linear span of S is $L(S)$, the set of all finite linear combinations of members of S. A finite linear combination of members of $V : x_1, x_2, \ldots x_n \in V$ is a vector $\sum_n a_i x_i$ where $a_i \in F$. By convention, let the linear span of the empty set $L(\phi) = \{0\}$ just contain the zero vector. Call this the zero-subspace.

Lemma 1D Let $S \subseteq V, L(S)$ is the smallest subspace of V containing S. That is, $L(S)$ is a subspace of V containing S, and if $S \subseteq U \subseteq V$, then $L(S) \subseteq U$, where U is a subspace.

Proof In HW2.

1.4 13 Jan 2014 Lecture

Definition A set of vectors in subset $S \subseteq V$ is linearly dependent if $\exists$ a non-empty finite subset $\{s_1, \ldots s_n\}$ of S and $a_i \in F, 1 \leq i \leq n$, no all 0, such that $O = a_1 s_1 + \ldots a_n s_n$. Call this a dependence relation on S.

Definition A set S is linearly independent if S is not dependent.

Example The empty set ϕ is independent, but the set containing the zero vector alone is dependent.

Lemma 1E Each subset of an independent set is independent

Proof Suppose we have an independent set S. We proceed by contradiction. Let $T \subseteq S$. If $\{t_1, \ldots t_n\} \subseteq T$ and $a_i \in F$, not all 0, with $\sum_i a_i t_i = 0$. This is a contradiction since this will also be a dependence relation in S, which is an independent set.

Theorem 1F Let S be an independent set, and $x \in V \backslash S$. Then $S \cup \{x\}$ is independent iff $x \notin L(S)$.

Proof It suffices to prove the contrapositive. $S \cup \{x\}$ is dependent iff $x \in L(S)$. For the $\Rightarrow$ case, assume $x \in L(S)$. Then $x = \sum_i = a_i s_i$ for some $s_i \in S, a_i \in F$. Then $1 \cdot x - \sum_i a_i s_i = 0$. But this is a dependence relation on $S \cup \{x\}$.

Definition: Generation and Basis A subset S of V generates V if $L(S) = V$. A basis for V is an independent generating set of V.

Lemma 1G Let $X = \{x_1, \ldots, x_n\} \subseteq V$. Then the following are equivalent: (1) X is a basis of V, and (2) each vector can be written uniquely as a linear combination of these vectors (up to reordering).

Proof (1) $\implies$ (2): Assume X is a basis of V. Then $V = L(X)$. For every vector $v \in V, \exists$ at least 1 expression $v = \sum_i a_i x_i$. Suppose $v = \sum_i b_i x_i$ also. Taking the difference, we have $\sum_i (a_i - b_i) x_i = 0$. But since X is independent, we require that $a_i - b_i = 0$, which shows that the coefficients are unique.

1.5 15 Jan 2014 Lecture

Definition: Order Let X be a set. Define the order of X to be: $|X| = \begin{cases} \infty & \text{if X is infinite} \\ n & \text{if X is finite with exactly n elements} \end{cases}$. The empty set has order zero.

Lemma 1H Let Y be an independent subset of V and X a generating set for V with $Y \subseteq X$. Then $\exists$ a basis B with $Y \subseteq B \subseteq X$.
Proof Proof when X is finite (infinite case requires the axiom of choice): We want to pick B so that (i)$Y \subseteq B \subseteq X$, (ii) B is independent, and (iii)$|B|$ to be maximal. This choice is possible because Y satisfies (i) and (ii), so there exists a B satisfying (i) and (ii). Now $|X| < \infty$, so for all B that satisfies (i) and (ii) then $|B| \leq |X| < \infty$. Now we want to show that B is a basis for V. By (ii), B is independent, so it remains to show that $V = L(B)$. Suppose $X \subseteq L(B)$. Then $V = L(X) \subseteq L(B)$, latter by problem 2 on homework 2. Now since B is in V and V is a linear space, addition and multiplication of elements of B is closed in V. Hence we have $L(B) \subseteq V$. Combining, $V = L(B)$. Hence it is OK in this case. Now we assume that $X \not\subseteq L(B)$ and we need to produce a contradiction. Then $\exists x \in X \backslash L(B)$. Let $A = B \cup \{x\}$. As $x \notin L(B)$ and as B is independent, by Lemma 1F, A is independent. Also, $B \subseteq X$ and $x \in X$ then $A \subset X \implies A$ is a set that satisfies (i) and (ii). But $|A| = |B| + 1 > |B|$, contradicting the requirement (iii) that $|B|$ is maximal. Contradiction. Hence $X \subseteq L(B)$.

Corollary 1I (1) V has a basis. (2) Each generating set in V contains a basis. (3) Each independent subset of V is contained in a basis.

Proof (1) Apply Lemma 1H with Y being the empty set and $X = V$. The empty set is independent. Visibly, $V = L(V)$. Then by Lemma 1H, there exists a basis B. (2) Let X generate V. Then apply Lemma 1H to this choice of X and Y being the empty set. Then Lemma 1H says that there exist a basis $B \subseteq X$. (3) Let $Y \subseteq V$ be independent. Apply Lemma 1H to this Y and $X = V$ to get basis B with $Y \subseteq B$.

Lemma 1J: Replacement Lemma Let X be a basis for V, $Y \subseteq V$ independent and $y \in Y \backslash X$. Then there exists $x \in X \backslash Y$ such that $(X \backslash \{x\}) \cup \{y\}$ is a basis for V.

Theorem 1.6 All bases for V has the same order.

Proof If all bases are infinite then all bases have order ∞. So we can assume that V has a finite basis Y. (i) Pick Y if smallest possible order (possible due to the finite size of Y). (ii) Pick basis X so that $|X| \neq |Y|$ and $|X \cap Y|$ is maximal subject to this constraint. We can assume the theorem fails, so there exists a basis X with $|X| \neq |Y|$. The maximal choice is possible as $|X \cap Y| \leq |Y| \leq \infty$. Suppose $|Y| \neq |X|$, then $\exists x \in X \backslash Y$. Since X is a basis, it is independent. Since $Y \subseteq X \implies Y \cup \{x\} \subseteq X$. By Lemma 1E, $Y \cup \{x\}$ is independent. On the other hand, Y is a basis $\implies V = L(Y) \implies x \in L(Y) \implies Y \cup \{x\}$ is dependent. So $Y \not\subseteq X$. Pick $y \in Y \backslash X$. By Lemma 1J, $\exists x \in X \backslash Y$ such that $B = (X \backslash \{x\}) \cup \{y\}$ is a basis for V. $|B \cap Y| = |X \cap Y| + 1 > |X \cap Y|$ so by (ii), $|B| = |Y|$. By construction, $|B| = |X| \implies |X| = |B| = |Y|$ contrary to (ii).

Definition: Dimension The dimension of V is the order of a basis of V.

Theorem 1K Let $\dim(V) = n < \infty$ and $X \subseteq V$. Then the following are equivalent: (1) X is a basis, (2) X is independent and $|X| = n$ (3) X generates V and $|X| = n$. Hence if a set has the right order, then we only need to check that it is either independent or generates V to conclude that it is a basis.

Proof We need to show that each statement implies the other. It will suffice to show 1 $\implies$ 2 and 3. and 2 $\implies$ 1 and 3 $\implies$ 1. 1 $\implies$ 2 and 3 As X is a basis, by definition, X is independent and generates V. By definition of dimension, $|X| = \dim(V) = n$. For 2 $\implies$ 1, assume X is independent and of order n. By Cor 1I, $x \subseteq B$ a basis of V. But $|B| = n = |X| < \infty$ and $X \subseteq B \implies B = X$ hence X is a basis. For 3 $\implies$ 1, assume X generates V and has order n, by Cor 1I, then there exists a basis B contained in X. Again, $|B| = n = |X| \implies B = X$.

1.6 Recitation 16 Jan 2014

Example Let V be a F-vector space of dim n. Prove that $V \cong F^n$, or that V is isomorphic to F^n.

1.7 Lecture 17 Jan 2014

Definition: Linear Transformation/Map Let U and V be vector spaces over F. A linear transformation from U to V is the function $f : V \to V$ which is linear. Linear: f preserves addition and scalar multiplication:

$$f(x + y) = f(x) + f(y)$$
$$f(ax) = af(x), \forall x, y \in U, a \in F.$$

Equivalently: $f(ax + by) = af(x) + bf(y)$.

Lemma 2A Let $f : U \to V$ be linear. Then the following are true:

1. $f(O) = O$
2. $f(-u) = -f(u), u \in U$
3. $f(\sum_i a_i u_i) = \sum_i a_i f(u_i)$
4. If W is a subspace of U, then $f(W)$ is a subspace of V.
5. For every subset $S \subseteq U, f(L(S)) = L(f(S))$.

Proof (1) $f(O) = f(O + O) = f(O) + f(O) \implies f(O) = O$

(2) By linearity of scalar multiplication

(3) By repeated application of linearity.

Theorem 2.12 Let X be a basis for U and $f_x : X \to V$ a function. Then (1) $\exists$ a unique linear map $f : U \to V$ extending f_x. Extending: $f(x) = f_x(x), x \in X$. Or you can say that the restriction of f onto vectors in X is f_x. (2) f is defined by $f(\sum_i a_i x_i) = \sum_i a_i f(x_i), x_i \in X$.

Proof Since X is a basis for U, any vector in U can be written uniquely as a linear combination of the basis vectors in X: $u = \sum_i a_i x_i$ (Lemma 1G). Define $f : U \to V, u \mapsto \sum_i a_i f_x(x_i)$. We check that f is linear. Let $w = \sum_i b_i x_i \in U$. Then $f(cu + dw) = f(c\sum a_i x_i + d\sum b_i x_i) = f(\sum(ca_i + db_i)x_i) = \sum(ca_i + db_i)f(x_i) = c\sum a_i f(x_i) + d\sum b_i f(x_i) = cf(u) + df(w)$. We prove that this linear map is unique: Suppose $g : U \to V$ is another linear extension of f_x. Then $g(u) = g(\sum a_i x_i) = \sum a_i g(x_i) = \sum a_i f_x(x_i) = f(u)$.

Defintiion: Space of Linear Maps Define $\mathcal{L}(U, V)$ to be the set of all linear maps $f : U \to V$. Make $\mathcal{L}$ into a vector space by defining vector addition and scalar multiplication: $(f + g)(u) = f(u) + g(u), (af)(u) = af(u), f, g \in \mathcal{L}(U, V), u \in U, a \in F$. Zero vector in $\mathcal{L}$ is a the zero-map: function that maps every element of U to the zero element of V. Every linear transformation can be represented by a matrix. Assume that dim(U) is n, and dim(V)=m, $n, m \in \mathbb{Z}^+$. Pick an ordered basis $X = \{x_1, x_2, \ldots x_n\}$ for U and $Y = \{y_1, y_2, \ldots y_m\}$ for V. Let f be a function in $\mathcal{L}(U, V)$. Then we know that $f(x_i) \in V$. But every vector in V is a linear combination of the basis elements in V. Then there exists a unique choice of $a_{ij} \in F$ such that we can represent the value of $f(x_i)$ uniquely as a linear combination $f(x_i) = \sum_{j=1}^m a_{ij} y_j$. Then (a_{ij}) is the matrix representing the linear combination f. Define the matrix of f with respect to the bases X and Y to be the $m \times n$ matrix $(a_{ij}) \in M_{m,n}$. Write $M_{X,Y}(f) = (a_{ij})$.

1.8 Lecture 22 Jan 2014

Special case Consider $\mathcal{L}(V)$ for $\mathcal{L}(V,V)$, maps from V to V. Given some linear mapping $f \in \mathcal{L}(V)$ and X an ordered basis for V, write $m_X(f)$ for the matrix $m_{X,X}(f)$.

Example Consider the identity map $id_V \in \mathcal{L}(V)$. Then $id_V(x_j) = x_j \implies m_X(id_V) = (\delta_{ij})$, where δ_{ij} is the Kronecker delta.

Isomorphism of vector spaces An isomorphism from U to V is a 1-1 correspondence $f : U \mapsto V$ which is linear. We say that U is isomorphic to V if $\exists$ an isomorphism from U to V. Write $U \cong V$ to indicate that U is isomorphic to V.

1-1 correspondence $\forall v \in V, \exists! u \in U$ s.t. $f(u) = v$.

Theorem 2.15 Let U be n-dimensional with basis X and V be m-dimensional with basis Y. Then $m_{X,Y} : \mathcal{L}(U,V) \mapsto M_{m,n}$, and $m_{X,Y}$ is an isomorphism.

Proof Let $M = m_{X,Y}$. Pick $f, g \in \mathcal{L}(X,Y), c, d \in F$. Set $M(f) = (a_{ij})$ and $M(g) = (b_{ij})$. We need to prove that M is linear and a 1-1 correspondence. Check that M is linear by looking at a linear combination of the functions operating on a basis vector x_j: $(cf + dg)(x_j) = cf(x_j) + dg(x_j) = c\sum_i a_{ij}y_i + d\sum_i b_{ij}y_i = \sum_i (ca_{ij} + db_{ij})y_i$. Then $M(cf + dg) = c(a_{ij}) + d(b_{ij}) = cM(f) + dM(g)$. 1-1 correspondence as an exercise.

Theorem 2.16 Let V be an n-dimensional F-space with basis X and $f, g \in \mathcal{L}(V)$. Then (1) $f \circ g \in \mathcal{L}(V)$, (2) $m_X(f \circ g) = m_X(f) \cdot m_X(g)$, where we use matrix multiplication in M_n.

Proof (1) For $u, v \in V, a, b \in F$, $(f \circ g)(au + bv) = f(g(au + bv)) = f(ag(u) + bg(v)) = af(g(u)) + bf(g(v)) = a(f \circ g)(u) + b(f \circ g)(v)$ hence $(f \circ g)$ is linear. (2) Let $M = m_X, M(f) = (a_{ij}), M(g) = (b_{ij})$, X be an ordered basis. Then $(f \circ g)(x_j) = f(g(x_j)) = f(\sum_k b_{kj}x_k) = \sum_k b_{kj}f(x_k) = \sum_k b_{kj}\sum_i a_{ik}x_i = \sum_i (\sum_k a_{ik}b_{kj})\, x_i = \sum_i c_{ij}x_i$ where $c_{ij} = \sum_k a_{ik}b_{kj}$. Hence $M(f \circ g) = (c_{ij}) = M(f) \cdot M(g)$. We observe that M preserves multiplication.

Null space/kernel Let $f : U \mapsto V$ be linear. The nulll space of f is the set of all elements in the domain space which map to zero $\{u \in U : f(u) = O\}$. Write $N(f)$ for the null space of f.

Theorem 2.2 $N(f)$ is a subspace of U.

Proof We check that $N(f)$ is nonempty and is closed under addition and scalar multiplication. The zero vector is in $N(f)$ since linear maps always map the zero vector of the domain to the zero vector of the range. Hence $N(f)$ is non-empty. Let $x, y \in N(f)$ and $a, b \in F$. Consider the linear combination $f(ax + by) = af(x) + bf(y) = a \cdot O + b \cdot O = O$. Hence $ax + by \in N(f)$ also.

Theorem 2B Let X be a basis for U and $f : U \mapsto V$ be linear. Then the following are equivalent: (1) f is an isomorphism (2) $N(f) = 0$ and $f(x)$ is a basis for V. (3) $N(f) = 0$ and $f(U)$ contains a basis for V.

Proof $1 \implies 2$: Assume f is an isomorphism. Then f is a 1-1 correspondence. Let $x \in N(f)$. We need to show that $x = O$. We know that $f(x) = O = f(O)$ because the zero vector always maps to the other zero vector. But since f is a 1-1 correspondence, the only vector x that can satisfy this is $x = O$. Hence $N(f) = O$. As f is a 1-1 correspondence, $f(U) = V$. Then $V = f(U) = f(L(X)) = L(f(X))$ by Lemma 2A.5. Then $f(X)$ generates V. Now we need to show that $f(X)$ is independent to show that it is a basis for V. Suppose $b_1, \ldots b_n \in F$ with $O = \sum_i b_i f(x_i) \implies \sum_i b_i x_i \in N(f)$. But x_i is independent since it is a basis. Hence $b_i = 0 \forall i \implies \{f(x_i)\}$ is independent.

1.9 Recitation 23 Jan 2014

Sum of subspaces Let U and W be subspaces of a vector space V. We can define U+W to be the smallest subspace containing U and W. But by HW2 Q2(b), we know that this will be the linear span of the intersection. $U + W = L(U \cup W)$.

Lemma Let S be a spanning set for U, T be a spanning set for W. Then $S \cup T$ span U+W.

Proof By HW2Q2(d), we have that $U = L(S) \subseteq L(S \cup T)$ and $W = L(T) \subseteq L(S \cup T)$. Hence $U \cup W \subseteq L(S \cup T)$. Now since $L(S \cup T)$ is a subspace $U \cup W \subseteq L(S \cup T) \implies L(U \cup W) \subseteq L(S \cup T) \implies U + W \subseteq L(S \cup T)$. Clearly, $S \cup T \subseteq U \cup W \subseteq U + W$, so U+W is a subspace, by $L(S \cup T) \subseteq U + W$. Hence $L(S \cup T) = U + W$.

Intersection/Sum Dimension Theorem $\dim(U+W)+\dim(U\cap W)=\dim(U)+\dim(W)$.

Proof We seek the basis for $U+W$ in terms of $U\cap W, U$ and W. Let B be a basis for $U\cap W$. Then by Corollary 1I, we can extend this into a basis for U and another for W. Call $B\cup B_1$ a basis for U, and $B\cup B_2$ a basis for W. We claim that $B\cup B_1\cup B_2$ is a basis for U+W. To check that this is a basis, we need to show that this is independent and it spans U+W. Now we know that, from Lemma, that the union of the two spanning sets span U+W. So we only need to check that $B\cup B_1\cup B_2$ is linearly independent. Consider a linear combination of the elements in this combined basis set which gives zero: $\sum\alpha_i(B)_i+\sum\beta_j(B_1)_j+\sum\gamma_k(B_2)_k=0$. Rearranging, $\sum\alpha_i(B)_i+\sum\beta_j(B_1)_j=\sum-\gamma_k(B_2)_k$. Call the LHS v. Then since v is a linear combination of elements in $B\cup B_1$, we know that $v\in L(B\cup B_1)=U$. From the right hand side, we know that $v\in L(B_2)\subseteq L(B\cup B_2)=W$. Hence we know that v lies in the intersection $v\in U\cap W$. Hence we can write v as a linear combination of elements in B, which is the basis for $U\cap W$. But this expression is unique, since B is a basis. Hence $\beta_j=0$ for all j, since $\sum\alpha_i(B)_i$ is already a linear combination of elements in B. Hence we have that $\sum\alpha_i(B)_i=\sum-\gamma_k(B_2)_k=v$. Now we know that $B\cup B_2$ is a linear independent, since it is a basis for W. Hence we require $\alpha_i,\gamma_k=0$ for all i,k. Hence $B\cup B_1\cup B_2$ is linearly independent, and is a basis for U+W. Now considering the dimensions, $\dim(U+W)+\dim(U\cap W)=|B\cup B_1\cup B_2|+|B|=2|B|+|B_1|+|B_2|=(|B|+|B_1|)+(|B|+|B_2|)=|B\cup B_1|+|B\cup B_2|=\dim(U)+\dim(W)$.

1.10 Lecture 24 Jan 2014

Note Suppose $f:A\to B$ is a 1-1 correspondence, and suppose $X\subseteq A$. Then (a) $f:X\to f(X)$ is also a 1-1 correspondence and (b) $|A|=|B|$.

Remark 1 Suppose $U\cong V$, then $\dim(U)=\dim(V)$.

Proof Let $f:U\to V$ be an isomorphism. Let X be a basis for U. By Lemma 2B, $f(X)$ is a basis for V (i.e. $\dim(V)=|f(X)|$. As $f:U\to V$ is a 1-1 correspondence, then $f:X\to f(X)$ is also a 1-1 correspondence. Hence we have that $|X|=|f(X)|$. Now $\dim(U)=|X|=|f(X)|=\dim(V)$.

Theorem 2.3 Let $f:U\to V$ be linear and $\dim(U)<\infty$. Then $\dim(U)=\dim(f(U))+\dim(N(f))$, where $N(f)$ is the null space of f.

Proof By Lemma 2A.4, $f(U)\leq V$ (f(U) is a subspace of V). Also as $f:U\to V$ is linear, $f:U\to f(U)$ is linear. Hence replace V by f(U), and WLOG we may assume that V=f(U). Let n=dim(U), m=dim(N(f)), and let Y be a basis for N(f). Now Y is independent. Hence by Corollary 1I (every independent set is contained in some basis), there exists a basis X of U with $Y\subseteq X$. Choose X so that Y is the first m elements of X. Let Z be the remaining elements in X not contained in Y, and W=L(Z). Z generates W and is independent, hence Z is a basis for W. Hence $\dim(W)=|Z|=n-m$.

Define $g:W\to U$ to be the restriction of f to subspace W. As f is linear, its restriction g is also linear. Claim 1: V=g(W), i.e. V is the image of W under g, so g(W) contains a basis of V. Proof of Claim 1: Let $v\in V$. As $V=f(U),\exists u\in U$ with $f(u)=v$. Write $u=\sum a_ix_i$. As $Y\subseteq N(f)$, $f(x_1)=0,\forall 1\leq i\leq m$, by defintion of the null space. Hence, $v=f(u)=f(\sum a_ix_i)=\sum a_if(x_i)$. We now separate the summation: $\sum_{i=1}^n a_if(x_i)=\sum_{i=1}^m a_if(0)+\sum_{i=m+1}^n a_if(x_i)=\sum_{i>m}a_if(x_i)=f(\sum_{i>m}a_ix_i)=g(\sum_{i>m}a_ix_i)\in g(W)$ since $\sum_{i>m}a_ix_i\in W$. Claim 2: N(g)=0.

As statements (1) and (3) of Theorem 2B are equivalent, we conclude that $g:W\to V$ is an isomorphism. Now $\dim(V)=\dim(W)=n-m=\dim(U)-\dim(N(f))$.

Theorem 2C Let $f:U\to V$ be linear and assume $\dim(U)=\dim(V)=n<\infty$. Then the following are equivalent: (1) f is an isomorphism (2) N(f)=0, (3) f(U) contains a basis of V.

Proof We observe that $1\implies 3$, since Theorem 2B states the necessary and sufficient conditions for an isomorphism. $(3)\implies(2)$:. Assume that X is a basis for V contained in $f(U)$. As X is a basis for V, $V=L(X)\subseteq f(U)\subseteq V$ by HW2, since $f(U)\leq V$ and $L(X)$ is the smallest subspace of V containing X. Hence since $f(U)$ is trapped between two Vs, we have that $V=f(U)$. Now by Theorem 2.3, $\dim(N(f))=\dim(U)-\dim(f(U))=n-n=0$. But the only subspace with zero dimension is the null vector. Hence $N(f)=0$, so (2) is true. Prove $(2)\implies(1)$ as an exercise.

Corollary 2D Two finite dimensional vector spaces on F are isomorphic iff they have the same dimension.

Proof $\implies$: We know this to be true by Remark 1. $\Leftarrow$: Assume $\dim(U)=\dim(V)=n$. Let $X=\{x_1,\dots x_n\}$ and $Y=\{y_1,\dots,y_n\}$ be bases for U and V respectively. Define the function $f_X:X\to V\subseteq V$ by $f_X(x_i)=y_i$. By Theorem 2.12, there exists a unique linear map $f:U\to V$ extending f_X. Then $f(X)=Y$ is a basis for V. Hence by Theorem 2C(3), f is an isomorphism.

Corollary 2E Each n-dimensional F-space is isomorphic to $V_n(F)$, the vector space of ordered n-tuples.

Proof Now we know that $\dim(V_n) = n$, so we can apply Corollary 2D.

Inverses of functions Let X and Y be two sets and $f : X \to Y$. An inverse for f is a function $g : Y \to X$ which satisfies $g \circ f = id_X$, and $f \circ g = id_Y$. Equivalently, $g(f(x)) = x \forall x \in X$ and $f(g(y)) = y \forall y \in Y$.

Caution when using Apostol Inverses: Ch2 Section 6 talks about left inverses and right inverses for f. But these inverses are functions $g : f(X) \to X$ rather than from Y to X.

Lemma 2F Let $f : X \to Y$ be a function. Then (1) f has an inverse iff f is a 1-1 correspondence, (2) If f has an inverse, that inverse is unique, (3) If f has an inverse, then $f^{-1} : Y \to X$ is a 1-1 correspondence with $(f^{-1})^{-1} = f$.

Proof We proved (1) and (2) when X=Y, and the same proof works here. Now we prove (3): by definition of the inverse function, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$, but the same equations show that f is an inverse function for f^{-1}. Now apply (1) and (2).

1.11 Lecture 27 Jan 2014

Lemma 2G Let $f : U \to V$ be an isomorphism. Then (1) f has an inverse function $f^{-1} : V \to U$. (2) f^{-1} is also an isomorphism.

Proof As f is an isomorphism, f is 1-1, so by 2F, it has an inverse f^{-1}. Also, by 2F, $f^{-1} : V \to U$ is a 1-1 correspondence. To prove (2), it remains to show that the inverse function is linear.

Algebra of matrices and linear maps Let n be a positive integer. Recall M_n is the F-space of all n by n matrices with entries in F. Also recall matrix multiplication is a binary operation on M_n which has an identity: $I = \text{diagonal}(1, 1, 1, \ldots)$ such that $I \cdot A = A \cdot I = A \forall A \in M_n$.

Theorem 2.17 Matrix multiplication is associative and distributive.

For the following statements, let $\dim(V) = n$, $X = \{x_1, \ldots x_n\}$ be a basis for V, $\mathcal{L}(V)$ is the F-space of linear maps in V. By Theorem 2.15, $m_X : \mathcal{L} \to M_n$ is an isomorphism. By Theorem 2.16 of $f, g \in \mathcal{L}(v)$, $m_X(f \circ g) = m_X(f) \cdot m_X(g)$.

Theorem 2H Let $m = m_X : \mathcal{L} \to M_n$ and let $f \in \mathcal{L}$ and $A = m(f)$. Then the following are true: (1) $m(id_V) = I$, (2) f has an inverse iff A has an inverse. (3) If f has an inverse, then $m(f^{-1}) = A^{-1}$. (4) $m^{-1}(AB) = m^{-1}(A) \circ m^{-1}(B) \forall A, B \in M_n$.

Proof (1) Done in example 6. (2) and (3): Suppose f^{-1} exists. Then $A \cdot m(f^{-1}) = m(f) \cdot m(f^{-1})$. By theorem 2.16, this is equal to $m(f \circ f^{-1}) = m(id_V) = I$. Similarly, $m(f^{-1}) \cdot A = I$. Hence $m(f^{-1})$ is an inverse of matrix A, and is unique. Use similar argument to part (4) to prove the other direction for (2). (4): By Theorem 2.15, $m : \mathcal{L} \to M_n$ is a 1-1 correspondence and by Theorem 2.16, it preserves multiplication. By Lemma 2F, m has an inverse function $m^{-1} : M_n \to \mathcal{L}$. Notice that as $m(f) = A$, then $m^{-1}(A) = f$. Let $g = m^{-1}(B)$. This means that $m(g) = B$. Then $m(f \circ g) = m(f) \cdot m(g) = A \cdot B$ by Theorem 2.16. Then $m^{-1}(AB) = m^{-1}(m(f \circ g)) = f \circ g = m^{-1}(A) \circ m^{-1}(B)$.

Change of coordinates By 2.15, $m_X : \mathcal{L} \to M_n$ is an isomorphism. Consider what happens when we change coordinates and choose a different basis Y. More precisely, if $f \in \mathcal{L}$ and Y is another basis for V, what is the relationship between $m_X(f)$ and $m_Y(f)$?

Theorem 4.6 Let $X = \{x_1, \ldots x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be bases of V and $f \in \mathcal{L}(V)$. Let g be the unique member of $\mathcal{L}(V)$ with $g(x_i) = y_i \forall 1 \leq i \leq n$. Then (1) $g : V \to V$ is an isomorphism, so $B = m_X(g)$ has an inverse B^{-1}. (2) $m_Y(f) = B^{-1} m_X(f) B$

Proof (1) Let $m = m_X$. By Theorem 2.12, there exists a unique $g \in \mathcal{L}(V)$ with $g(X) = Y$. As X is a basis and $g(X) = Y$ is also a basis, then g is an isomorphism by Theorem 2C. So g has an inverse g^{-1}. Now by Lemma 2G, $g^{-1} \in \mathcal{L}$ is also an isomorphism. Then by Theorem 2H, $B = m(g)$ also has an inverse, hence (1) is true. For (2), let $m(f) = (a_{ij})$. Then $f(x_j) = \sum_i a_{ij} x_i$. Let $B = (b_{ij})$ and $B^{-1} = (c_{ij})$. As $B = m(g)$, we have $B^{-1} = m(g^{-1})$, so $g(x_j) = \sum_i b_{ij} x_i$ and $g^{-1}(x_k) = \sum_i c_{ik} x_i$. Thus $x_k = g(g^{-1}(x_k)) = g(\sum_i c_{ik} x_i) = \sum_i c_{ik} g(x_i) = \sum_i c_{ik} y_i$. $f(y_j) = f(g(x_j)) = f(\sum_r b_{rj} x_r) =$

$\sum_r b_{rj}f(x_r) = \sum_r b_{rj}(\sum_k a_{kr}x_k) = \sum_{r,k,i} b_{ij}a_{kr}c_{ik}y_i = \sum_i d_{ij}y_i$ where $d_{ij} = \sum_{r,k} b_{rj}a_{kr}c_{ik} = \sum_{k,r} c_{ik}a_{kr}b_{rj}$ = ith, jth entry in $B^{-1}m(f)B$. Hence $m_X(f) = (d_{ij}) = B^{-1}m(f)B$.

Definition: Similarity Two square matrices $A, C \in M_n$ are similar if there exists an invertible matrix $B \in M_n$ such that $C = B^{-1}AB$.

Theorem 4.8 Let $A, C \in M_n$. Then the following are equivalent: (1) A and C are similar. (2) There exists bases X and Y of V and $f \in \mathcal{L}$ such that $m_X(f) = A$ and $m_Y(f) = C$.

Proof (2) $\implies$ (1). Assume (2) holds. Let $g \in \mathcal{L}$ with $g(x_i) = y_i \forall i$. Then $C = m_Y(f) = B^{-1}m_X(f)B = B^{-1}AB$.

1.12 Lecture 29 Jan 2014

Definition: Transpose Let $A = (a_{ij}) \in M_{m,n}$. The transpose of A is the matrix $A^t \in M_{n,m}$ defined by $A^t = (a^t_{ij})$, where $a^t_{ij} = a_{ji}$.

Notation Define A_i to be the ith row of A. Define $A^{(j)}$ to be the transpose of the jth column (so it becomes a row) of A. The jth column is $A^{(j)t}$. The column space of A is the subspace $L(A^{(1)}, A^{(2)}, \ldots, A^{(n)})$ of V_m (since each vector has m components) spanned by the transpose of the columns of A. Define the rank of A to be the dimension of the column space. Write this as rk(A).

Systems of Linear Equations A system of linear equations in n unknowns $x_1, x_2, \ldots x_n$ is a system $\mathcal{F} = (\mathcal{F}_i, 1 \leq i \leq m)$ of m equations, where each equation $\mathcal{F}_i$ is $\sum_{j=1}^n a_{ij}x_j = b_i, a_{ij}, b_i \in F$ in the unknowns $x_1, \ldots x_n$. A solution to $\mathcal{F}$ is the vector $v = (v_1, \ldots v_n) \in V_n$ such that $\forall i, 1 \leq i \leq m, \sum_{j=1}^n a_{ij}v_j = b_i$ in F. Write $S(\mathcal{F})$ for the set of all solutions to $\mathcal{F}$. By HW1 Problem 2, $S(\mathcal{F})$ is a subspace of V_n iff $\mathcal{F}$ is homogenous. Homogenous: $b_i = 0, \forall i$. If E is a homogenous system, call $S(E)$ the solution space of E. Note that in a homogenous system E the zero vector is a solution to E. Write $E(A)$ to be the homogenous system of A. Each $\mathcal{F}_i$ corresponds to the row vector $A_i(\mathcal{F}) \in V_n$. Call $A(\mathcal{F})$ the matrix of $\mathcal{F}$. Conversely, given matrix $A \in M_{m,n}$, we can assign the homogenous system $E(A)$ of m equations in n unknowns. Finally, we can associate to A the linear map $T_A : V_n \to V_m$ with $x \mapsto (Ax^t)^t$, noting that V_n and V_m are the vector spaces of row vectors.

Lemma 2K Let $(\mathcal{F}) : \sum_{j=1}^n a_{ij}x_j = b_i, 1 \leq i \leq m$ be a system of linear equations. Set vector $b = (b_1, \ldots, b_m) \in V_m$. Let $A = A(\mathcal{F}) = (a_{ij}) \in M_{m,n}, E = E(A)$ and $S = S(E)$. Then the following are true: (1) $v \in S(\mathcal{F}) \iff T_A(v) = b$. (2) The image of T_A is the column space of A. (3) $N(T_A) = S$. (4) $n = rk(A) + \dim(S)$. (5) $\mathcal{F}$ has a solution iff b is contained in the column space of A.

Proof (1) Let $T = T_A$. Let $v = (v_1, \ldots v_n) \in V_n$. By definition of T, the following statements are true: $T(v) = (Av^t)^t = y = (y_1, \ldots y_n)$ where $y_i = \sum_{j=1}^n a_{ij}v_j$. In particular, $T(v) = b$ iff $\forall i, b_i = y_i = \sum_{j=1}^n a_{ij}v_j$ which is exactly what it means for v to be a solution $v \in S(\mathcal{F})$. Hence (1) is true. (3) Similarly, $v \in N(T)$ iff $T(v) = 0$ iff $\forall i, 0 = y_i = \sum_{j=1}^n a_{ij}v_j$ iff $v \in S(E) = S$. (2): By definition of T, $T(v) = (\sum_j a_{1j}v_j, \ldots, \sum_j a_{mj}v_j) = \sum_j v_j(a_{1j}, \ldots, a_{mj}) = \sum_j v_jA^{(j)} \in L(A^{(1)}, \ldots A^{(n)}) = C$, where C is the column space of A. So $T(V_n) \subseteq C$. Take $v = e_k$ = kth coordinate vector. That is $v_j = \begin{cases} 0 & ifj \neq k \\ 1 & ifj = k \end{cases}$. Then $T(v) = \sum_j v_jA^{(j)} = A^{(k)} \implies T(v) = A^{(k)} \in C$, that is $\forall k, T(e_k) \in C \implies T(V_n) = T(L(B))$, where $B = \{e_1, \ldots e_n\}$. $A^{(k)} \in T(v)$ is a subspace of $V_m \implies C = L(A^{(1)}, \ldots A^{(n)}) \in T(V) \implies$ (2) holds by HW2. (5) Note that $\mathcal{F}$ has solution iff $T(V_n) = C$ by (1) and (2). (4): $\dim(T(V_n)) = \dim(C) = rk(A)$ by definition of $rk(A)$. Since $n = \dim(V_n) = \dim(T(V_n)) + \dim(N(T))$ by Theorem 2.3. Hence $n = rk(A) + \dim(S)$.

Corollary 2L Let E be a system of m homogenous equations in n unknowns with solution space S. Then $\dim(S) = n - rk(E)$, where $rk(E) = rk(A(E))$.

Proof By Lemma 2K(4), $n = rk(A(E)) + \dim(S)$.

Corollary 2M Let $n > m$ and E be homogenous, with m equations on n unknowns. Then E has a non-zero solution.

Proof Write $r = rk(E) = \dim(C)$. $C \leq V_m$ is the column space. Then $r = \dim(C) \leq \dim(V_m) = m$. But we also have that $\dim(S) = n - r \geq n - m > 0 \implies \dim(S) > 0$. Hence S cannot just be the zero vector. Hence S has non-zero vectors.

1.13 Recitation 30 Jan 2014

Isomorphism not unique Unless n=0. The isomorphism $V \cong V_n$, where V is a vector space of dim n over F is not unique, in order words, there exists f, g such that $f : V \to V_N$ and $g : V \to V_n$ with $f \neq g$. To fix an isomorphism, we pick a basis $\{e_1, \dots e_n\}$ of V and define $f : V \to V_n$ with $e_i \mapsto (0, \dots 1, \dots, 0)$, where the 1 is in the ith position. If $n > 0$ and F is infinite, there exists infinitely many bases for V. In other words, there exsits infinitely many different isomorphisms $V \cong V_n$.

Remark We can replace V_n with the space of column vectors of length n. Call this space C_n.

Elementary Row Operations (1) Multiply a row by a non-linear scalar. (2) Add a multiple of one row to another. (3) Swap 2 rows.

1.14 Lecture 31 Jan 2014

Definition: Coset Let V be a vector space over F. Let subspace $U \leq V$. The coset of $v \in V$ with respect to the subspace U is $\{u + v : u \in U\} \subseteq V$. Write $U + v$ for this coset. Basically you are adding v to all the elements of U.

Remarks (1) Ths coset U+v is not usually a subspace. It is a subspace of V iff $v \in U$. (2) $w \in U + v$ iff $w - v \in U$ (obviously).

Theorem 2.19 Let $(\mathcal{F}) : \sum_{j=1}^{n} a_{ij} x_j = b_i$ be a system of linear equations with matrix $A = (a_{ij})$. Let $S = S(E)$ be the solution space of the homogeneous system $E = E(A)$. Suppose $t = (t_1, \dots, t_n)$ is a solution to $\mathcal{F}$. Then $S(\mathcal{F}) = S + t$ is a coset.

Proof Let $v = (v_1, \dots, v_n) \in V_n$. We have to show that $v \in S(\mathcal{F})$ iff $v \in S + t$. By Remark (2), this is the same thing as showing $v \in S(\mathcal{F}) \iff v - t \in S$. But $v - t \in S \iff \forall i, 0 = \sum_j a_{ij}(v_j - t_j) = \sum_j a_{ij} v_j - \sum_j a_{ij} t_j$. But t is a solution to $\mathcal{F}$. Hence we can write this as $\sum_j a_{ij} v_j - b_i = 0 \iff \sum_j a_{ij} v_j = b_i \iff v \in S(\mathcal{F})$.

Summary (a) If $\mathcal{F}$ has a solution t then $S(\mathcal{F}) = S + t$ is a coset. (b) $\dim(S) = n - rk(A)$ by Corollary 2L.

Theorem 2N Assume $\mathcal{F}$ is a system with n equations and n unknowns and assume $A = A(\mathcal{F})$ is invertible. Then (1) $\mathcal{F}$ has a unique solution $S(\mathcal{F})$. (2) $s^t = A^{-1} b^t$, where $b = (b_1, \dots b_n)$.

Proof Let $v = (v_1, \dots v_n) \in V_n$. By Lemma 2K.1, $v \in S(\mathcal{F}) \iff b = T_A(v) = (Av^t)^t$. This equation holds iff $b^t = Av^t$. Multiply this on the left by the inverse of A: $A^{-1} b^t = A^{-1} A v^t = v^t$. This shows that $A^{-1} b^t$ is a solution.

Definition: Augmented matrix Define the augumented matrix of a system of linear equations to be $B(\mathcal{F}) \in M_{m,n+1}$, where B is obtained by adjoining b^t to A as the $(n + 1)$st column of B.

Lemma 2P If $\mathcal{F}'$ is equivalent to $\mathcal{F}$ then $S(\mathcal{F}') = S(\mathcal{F})$, where $\mathcal{F}'$ is obtained from $\mathcal{F}$ through a sequence of elementary row operations.

Proof It suffices to show that the elementary row operations do not affect the solution space of $\mathcal{F}$. We just consider the operation where one row is added to another. Let $B' = B(\mathcal{F}')$. Then $\forall j \neq k, B'_j = B_j$, while $B'_k = B_k + B_i$. Let $s = (s_1, \dots, s_n) \in S(\mathcal{F})$. This is a solution to $\mathcal{F}_j = \mathcal{F}'_j, \forall j \neq k$. So to show $s \in S(\mathcal{F}')$ must show s is a solution to $\mathcal{F}'_k$. We write $\sum_j a'_{kj} s_j = \sum_j (a_{kj} + a_{ij}) s_j = \sum_j a_{kj} s_j + \sum_j a_{ij} s_j = b_k + b_i = b'_k$. Hence $S(\mathcal{F}) \subseteq S(\mathcal{F}')$. A similar argument shows that $S(\mathcal{F}') \subseteq S(\mathcal{F})$. Hence they are equivalent.

Definition: Upper triangular matrix A matrix $C \in M_n$ is said to be upper triangular if $\forall j < i, c_{ij} = 0$. That is, all entries below main diagonal are zero.

1.15 Lecture 03 Feb 2014

Midterm Review of Chapter 1 and 2 Stuff need to know: Chapter 1, Sections 1-9: Subspaces, Linear Independence, Linear Span, Bases, Dimensions, HW2 Q2-3 (Properties of Linear Span), HW1 Q4 (Determinants and Inverse of 2x2 matrices). Chapter 2, Sections 1-5 and 9-18: Linear Transformations/Maps, Matrices, Isomorphisms, Vector space of linear maps, Vector Space of Matrices, Isomorphism of linear maps: If X and Y are bases in U and V respectively, we have the matrix correspondence $m_{X,Y} : \mathcal{L} \to M_{m,n}$ is an isomorphism, Change of Coordinates (Theorem 4.6), Systems of Linear Equations, Gauss Algorithm, Transposes. From Rec Session: Polynomials, Intersection/Sum Dimension Theorem,

1.16 Lecture 05 Feb 2014

Lemma If $C, D \in M_n$ commute, then C commutes with all powers of D. I.e. $CD^k = D^k C, \forall 0 < k \in \mathbb{Z}$.

Proof We proceed by induction on k. $k = 1$ holds by hypothesis. Assume it holds for some k. Then we inspect $CD^{k+1} = C(D^k D) = (CD^k)D = (D^k C)D = D^k(CD) = D^k(DC) = D^{k+1}C$.

Lemma If $A, B \in M_n$ anticommute, then A^2 commutes with B.

Proof $A^2 B = A(AB) = A(-BA) = -A(BA) = -(AB)A = -(-BA)A = (BA)A = BA^2$.

Lemma If A and B are invertible, then so is AB.

Proof We prove that $(AB)^{-1} = B^{-1}A^{-1}$. We need to show that $(AB)(B^{-1}A^{-1}) = I$. Applying associativity, this is true. We also need to show that $(B^{-1}A^{-1})(AB) = I$, which is also true by associativity. Hence (AB) has an inverse.

Lemma If $A^3 = 0$, then $A - I$ is invertible.

Proof Claim: Let $B = -(A^2 + A + I)$ then $(A - I)^{-1} = B$. Note that $A^3 - 1 = (A - I)(A^2 + A + I)$. But by assumption, $A^3 = 0$. Hence we have that $-I = (A - I)(A^2 + A + I)$ and we have an expression for the inverse of A-I.

Incorrect lemmas If A and B are invertible, A+B may not be invertible (Counterexample: I and -I, and the zero matrix is not invertible). If A, B and A+B are invertible, A-B may not be invertible. (Counterexample: I and I, and the zero matrix is not invertible)

1.17 Recitation 06 Feb 2014

Definition A field F is algebraically closed if every polynomial in $F[x]$ (that is, that has coefficients in F) has a root in F, i.e. given any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $a_i \in F, \forall i, \exists r \in F$ such that $p(r) = 0$. Note that $\mathbb{R}$ is not algebraically closed, since the equation $x^2 + 1 = 0$ has no solution in $\mathbb{R}$. But $\mathbb{C}$ is algebraically closed.

Problem If we are given a linear map $f : V \to V$, does there exist a basis of V such that $m_Y(f)$ is diagonal? Fix a basis for V. By Theorem 4.8, there exists a basis Y for V such that $m_Y(f)$ is diagonal iff $m_X(f)$ is similar to a diagonal matrix. This means that there exists an invertible matrix B such that $B^-1 m_X(f)B$ is diagonal.

Example Show that the matrix $A = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$. is similar to a diagonal matrix over $\mathbb{C}$ but not over $\mathbb{R}$. Write out a general matrix, then multiply out $B^{-1}AB$, setting non-diagonal entries to zero. Notice that the diagonal entries may or may not be zero. Zero matrix is not an invertible matrix.

Diagonalizable matrix Not every matrix is diagonalizable over $\mathbb{C}$ even though C is algebraically closed. Over an algebraically closed field F (e.g. $\mathbb{C}$), any matrix is similar to one in Jordan canonical form (Jordon normal form). A matrix is in Jordan canonical form if it has elements along the main diagonal, with possibly 1s above the main diagonal. All other entries are zero. Each Jordan block must contain the same diagonal value.

Example The infinite dimensional polynomial ring does not obey $\dim W = \dim V \iff W = V$ if $W \leq V$ is a subspace. Let $\{1, x, x^2, \ldots\}$ be a basis for V. Then $\dim V = \infty$. Then consider $W = \{p(x) : a_0 = 0\}$, the subspace of all polynomials with constant term equals zero. We claim that $\{x, x^2, x^3, \ldots\}$ is a basis for W. Now to prove that it is a basis, we just need to check linear independence and generation. Now we know that the basis for W is a subset of the basis for V, which is linearly independent. Hence W is a linearly independent set. It also clearly spans W. But since the basis is an infinite set, we conclude that $\dim W = \infty$. In fact, $W \cong V$, if we consider the linear map $f : V \to W$, $f(P(x)) \mapsto xP(x)$. This is because the bases for V and W have the same cardinality.

Example Let V be a finite dimensional vector space, $W \leq V$ a subspace. Let $v \in V, v \notin W$. Let $f : V \to V$ be a linear map such that $f(W) \in W$ so f restricts to a linear map $f|_W : W \to W \in V$. Suppose $f(v) \notin W$ and $N(f(w)) = 0$. Let $U = L(W \cup \{v\})$. Show that $\dim f(U) = \dim W + 1$.

Solution By the rank nullity theorem $\dim N(f|_W) + \dim f(W) = \dim W$. Hence $0 + \dim f(W) = \dim W$. Since $f(v)$ is not in W, $f(W) \leq W$, so W is properly contained in $f(U)$. so $\dim f(U) > \dim W$. On the other hand, any $u \in U$ is a finite sum $u = a_1 w_1 + \ldots + a_k w_k + bv$, where w_i is a basis for W. Since v is not in W, we can extend the basis of W by one by adding v.

Hence U is spanned by the union of the basis for W and v. Hence $\dim U = \dim W + 1$. Now we have that $\dim f(U) \leq \dim U$. If u_i is a basis for U, then $f(U) = L(f(u_i))$. But we also know that $\dim f(U) > \dim W$. Combining these things, we get the result.

1.18 Lecture 07 Feb 2014

Definition Define the determinant function $\det : M_n \to F$. Investigate permuations first:

Permuations Let $I = \{1, \ldots, n\}$. A permutation of I is a one-one correspondence of the set to itself $s : I \to I$. Write S_n for the set of all permutations of I. S_n is the symmetric group on I. The number of permutations is $|S_n| = n!$.

Cycle Notation Each $s \in S_n$ can be written in cycle notation: $s = (a_1, \ldots, a_\alpha)(b_1, \ldots, b_\beta) \ldots (z_1, \ldots, z_\zeta)$. All the elements are the elements of I in some order. This indicates: $s(a_i) = a_{i+1}, 1 \leq i < \alpha$, and $s(a_\alpha) = a_1$ (cycles back). Same for $b, \ldots, z$. The term $(a_1, \ldots, a_\alpha)$ is a cycle of s. This cycle has length α, since it has α elements involved in a cycle.

Example 1 Let $n = 6$. Let $r = (2, 3, 5)(1, 6, 4)$ and $t = (1)(2, 5)(3)(4)(6)$ be members of S_6. Note that t fixes 1,3,4 and 6, since it maps these elements back to itself and t interchanges 2 and 5.

Definition Define s to be even if s has an even number of cycles of even length. Define s to be odd if s has an odd number of cycles of even length. In example 1, r is even (no cycles of even length) and t is odd (1 cycle of even length). A permutation like t with 1 cycle of length 2 and n-2 cycles of length 1 is called a transposition. All transpositions are odd.

Convention Usually we suppress cycles of length 1. We don't write down cycles of length 1. Hence t in Example 1 would be $t = (2, 5)$.

Definition The sign function $sgn : S_n \to \{1, -1\}$ by $sgn(s) = \begin{cases} +1 & \text{if s is even} \\ -1 & \text{if s is odd} \end{cases}$.

Definition Let $A = (a_{ij}) \in NM_n$. The determinant of A is $\det(A) = \sum_{s \in S_n} sgn(s) a_{1,s(1)} \cdots a_{n,s(n)}$. Note that there are $n!$ terms in the summation.

Notation Given $A = (a_{ij}) \in M_n$ and $1 \leq i \leq n$, write A_i for the ith row. Write $A(s) = sgn(s) a_{1,s(1)} \cdots a_{n,s(n)}$. Hence we can write $\det(A) = \sum_{s \in S_n} A(s)$. Given row vectors $B^1, \ldots B^n \in V_n$, write $[B^1, \ldots, B^n]$ for the matrix where the ith row is B^i.

Example 2 Take $n = 2$. So $S_2 = \{id, t\}$ where $id = (1)(2)$ and $t = (1, 2)$ is a transposition. Let $A = \begin{pmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{pmatrix} \in M_2$. Then $\det(A) = A(id) + A(t) = sgn(id)(a_{1,id(1)} a_{2,id(2)}) + sgn(t)(a_{1,t(1)} a_{2,t(2)}) = a_{11}a_{22} - a_{12}a_{21}$.

Recall Let $f, g : I \to I$. By Lemma 1B, f is a permutation iff f has an inverse function $f^{-1} : I \to I$ such that $f \circ f^{-1} = f^{-1} \circ f = id$. Also, f^{-1} is also a permutation, and $(f^{-1})^{-1} = f$.

Remark 1 If $f, g \in S_n$ then $f \circ g \in S_n$.

Proof $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Lemma 3A The sign function preserves multiplication: $sgn(f \circ g) = sgn(f) sgn(g)$.

Proof Ask jeff.

Lemma 3B If matrix A is triangular, then the determinant of A is $a_{11}a_{22} \ldots a_{nn}$ the product of the entries along the diagonal.

Proof Claim $A(s) = 0$ unless s is the identity permutation. Consider $s \neq id$. Then $\exists i \in I$ with $s(i) < i$. Hence as A is triangular, this means that $a_{i,s(i)} = 0$. This zero makes the whole product zero. Hence $\det(s) = a_{1,id(1)} \ldots a_{n,id(n)}$.

Lemma 3C det is homogenous in each row. That is, if the kth row of matrix A is multiplied by a scalar to get a new matrix B, then $\det B = a \det A$.

Proof Let $B = (b_{ij})$ and $A = (a_{ij})$. Then $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq k \\ a \cdot a_{kj} & \text{if } i = k \end{cases}$. So $B(s) = sgn(s) b_{1,s(1)} \cdots b_{k,s(k)} \cdots b_{n,s(n)} = aA(s)$. Hence $det(B) = \sum_s B(s) = a \sum_s A(s) = a \det(A)$.

Lemma 3D det is additive. In each row, if $\exists i \in I$ such that $B = [A_1, \dots A_{i-1}, B_i, A_{i-1}, A_n]$ and $C = [A_1, \dots, A_{i-1}, A_i + B_i, A_{i+1}, \dots A_n]$ then det(C)=det(A)+det(B).

1.19 Midterm Review 07 Feb 2014

Practice Problem 1 Let V be the space of all real functions continuous on $[-\pi, \pi]$. Let S be the subset of V consisting of f that satisfy the following properties: $\int_{-\pi}^{\pi} f(t)dt = 0$, $\int_{-\pi}^{\pi} f(t)\cos(t)dt = 0$, $\int_{-\pi}^{\pi} f(t)\sin(t)dt = 0$. (a) Let $T : V \to V$ be linear with $T(f)(x) \mapsto \int_{-\pi}^{\pi}(1+\cos(x-t))f(t)dt, \forall f \in V$. Prove that the image of T is finite dimensional. (b) Find the null space.

Solution 1 (a) Note that $T(f)(x) = \int_{-\pi}^{\pi}(1 + \cos x \cos t + \sin x \sin t)f(t)dt$. We can split this into $T(f)(x) = \int_{-\pi}^{\pi} f(t)dt + \cos x \int_{-\pi}^{\pi} \cos t f(t)dt + \sin x \int_{-\pi}^{\pi} \sin t f(t)dt = c_1 + \cos x c_2 + \sin x c_3$, since we realize that the constants c_1, c_2, c_3 depend on f. Hence we have that the image of T is spanned by the set $\{1, \cos x, \sin x\}$. We check linear independence: $a_1 + a_2 \cos x + a_3 \sin x = 0, \forall x$. We check for $x = 0$, and get $a_1 = -a_2$. We can also check for $x = \pi/2$, and get $a_1 = -a_3$. Choose $x = -\pi/2$, and get another equation. Solve to get $a_1 = a_2 = a_3 = 0$. Hence the set spans the image and is linearly independent, hence it is a basis for the image. Hence the dimension of the image is 3.

(b) The null space consists of elements which satisfy $\int_{-\pi}^{\pi}(1 + \cos(x - t))f(t)dt = 0$. By the expansion, for $T(f)(x) = 0$, we require that $c_1 = c_2 = c_3 = 0$ since the basis vectors are linearly independent. But it is precisely the elements of S that satisfy this condition. Hence the null space is the subspace S. Note that by the null-rank theorem, $\dim V = \dim N(T) + \dim T(V)$. But we know that $\dim T(V) = 3$, and the $\dim V = \infty$, since it is the space of all real functions. Hence $\dim N(T) = \infty$.

Problem 2 Let $T : V \to W$ be a linear map and suppose $N(T) = 0$. Prove that if $\{v_1, \dots v_k\}$ is a linear independent set in V, then $\{T(v_1), \dots, T(v_k)\}$ is also linearly independent in W.

Solution Assume that we have a linear combination $a_1T(v_1) + \dots + a_kT(v_k) = 0$. We want to prove that $a_1 = \dots = a_k = 0$. Since T is a linear map, we can re-write this as $T(\sum a_i v_i) = 0$. But we know that the null space is 0. Hence $\sum a_i v_i = 0$. But v_i are all independent, hence $a_i = 0, \forall i$. Note that only when $N(T) = 0$ that the image of the basis vectors under T is a basis for the image. This is because $\dim V = \dim N(T) + \dim T(V)$, and $\dim N(T) = 0$. Hence the linearly independent vectors under T form a basis.

Problem 3 Let V be the space of polynomials of degree ≤ 3 in $\mathbb{R}[x]$. Consider the differential operator $D : V \to V$. Consider a basis for V, $B = \{1, x, x^2, x^3\}$. Find $m_B(D)$. (b) Let $t \in \mathbb{R}$ and define $g_i(x) = (x + t)^{i-1}$. Let $\gamma = \{g_1, g_2, g_3, g_4\} = \{1, x + t, x^2 + 2tx + t^2 = t^3 + 3t^2x + 3tx^2 + x^3$. Now we have a unique linear transofrmation mapping $f_i \mapsto g_i$. The matrix for that transformation is formed from the columns of the new basis vector coordinates in the old basis.

Solution 3 We just need to observe the effect of the operator on each of the basis vectors. $D(1) = 0 = (0, 0, 0, 0)_B, D(x) = 1 = (1, 0, 0, 0)_B, D(x^2) = 2x = (0, 2, 0, 0)_B, D(x^3) = 3x^2 = (0, 0, 3, 0)_B$. We can merge these vectors (by combining the columns) into the matrix.

Inverse of an upper triangular matrix Change the sign of entries where the sum of the row and column is odd.

1.20 Lecture 12 Feb 2014

Notation for Chapter 4 Let V be a vector space over $\mathbb{F}$. Let $\mathcal{L} = \mathcal{L}(V)$ be the space of linear maps on V with multiplication defined by composition. Let $f \in \mathcal{L}$.

Definition: Eigenvalue An eigenvalue for $f \in \mathcal{L}$ is an element $a \in \mathbb{F}$ such that $\exists$ a non-zero vector $v \in V$ such that $f(v) = a \cdot v$. Note that a can be zero, but the eigenvector cannot be zero.

Definition: Eigenspace The eigenspace for eigenvalue $a \in \mathbb{F}$ on V with respect to f consists of all the vectors $E(a) = \{u \in V : f(u) = a \cdot v\}$. Thus, a is an eigenvalue for f iff $E(a) \neq 0$. That is, there is at least one vector that satisfies the eigenvalue equation.

Definition: Eigenvectors The non-zero members of $E(a)$ are called the eigenvectors for a.

Example Consider the eigenspace for the eigenvalue zero. Then $v \in E(0) \iff f(v) = 0 \cdot v = 0 \iff v \in N(f)$. Then $E(0) = N(f)$. The eigenspace associated with the zero eigenvalue is the null space.

Example Consider the eigenspace for the eigenvalue 1 for the identity function in V. Then we require that $\forall v \in V, f(v) = 1 \cdot v = v \implies V = E(1)$ so 1 is the unique eigenvalue for the identity map.

Lemma 4A Let $f \in \mathcal{L}$ and $a \in F$. Then $E(a) = N(a \cdot id_V - f)$.

Proof Let $g = a \cdot id_v - f \in \mathcal{L}$ i.e. as id and f are in $\mathcal{L}$, the combination g of id and f are also in $\mathcal{L}$. Let $v \in V$. Then $g(v) = (a \cdot id - f)(v) = a \cdot id(v) - f(v) = a \cdot v - f(v)$. So $v \in N(g) \iff 0 = g(v) = a \cdot v - f(v) \iff f(v) = av \iff v \in E(a)$.

Theorem 4.2 Let $a_1, \ldots, a_m$ be distinct eigenvalues for f. Let v_i be an eignevector for a_i. Then the set of eigenvectors is independent.

Proof See notes.

Polynomial function of matrices Let x be a symbol. A polynomial in x over $\mathbb{F}$ is a formal sum $f(x) = \sum_{i=0}^{m} a_i x^i$ for some $m \in \mathbb{N}$ and $a_i \in \mathbb{F}$. Formally, f is an infinite sequence indexed by the natural numbers such that $a_i = 0$ for all but a finite set of indices $i \in \mathbb{N}$. Call a_i the ith coefficient of f. Two polynomials are equal iff the coefficients are equal. The zero polynomial is the polynomial such that all its coefficients are zero. Define the degree of the zero polynomial to be zero. If $f(x) \neq 0$ then the degree of f is defined to be $\max\{i : a_i \neq 0\}$. Write $F[x]$ for the set of all polynomials in x over $\mathbb{F}$. We define addition to be the addition of individual coefficients. We define multiplication to be the convolution $\sum a_i x^i \sum b_i x^i = \sum c_i x^i$ where $c_i = \sum_{j=1}^{k} a_j b_{k-j}$. Define multiplication by scalar to be the multiplication of each coefficient by that scalar. Now we have the $F[x]$ fulfills the axioms of a vector space over $\mathbb{F}$. Let $X = \{x^i, i \in \mathbb{N}\}$ be a basis for $F[x]$.

1.21 Recitation 13 Feb 2014

Rewriting cycles We can write a cycle $(a_1, \ldots, a_l) = (a_1, a_l)(a_1, a_{l-1}) \ldots (a_1, a_2)$, when composition starts from the right. Any $s \in S_n$ can be writte as a composition of transpositions. The sign is 1 is it can be written as a composition of an even number of transpositions. Sign is -1 if s can be written as a composition of an odd number of transpositions.

Cofactor Expansion Define the (i,j)th minor of $n \times n$ matrix A to be the matrix with the ith row and jth column deleted. Call this A_{ij}. Define the (i,j)th cofactor of A as $(-1)^{i+j}\det(A_{ij})$. Then the determinant of A is $\det(A) = \sum_{j=1}^{n} a_{kj}(-1)^{k+j}\det(A_{kj})$, or the sum of the cofactors along a row. Note that we can expand along a column too: $\det(A) = \sum_{j=1}^{n} a_{jk}(-1)^{k+j}\det(A_{jk})$.

1.22 Recitation 20 Feb 2014

Lemma If λ is an eigenvalue for $f : V \to V$, and if f is invertible, λ^{-1} is an eigenvalue of $f^{-1} : V \to V$.

Proof λ is an eigenvalue iff $\exists v \in V, v \neq 0 s.t. f(v) = \lambda v$. Hence $v = f^{-1}(\lambda v)$ and $\lambda^{-1} v = \lambda^{-1} f^{-1}(\lambda v) = f^{-1}(v)$. Hence λ^{-1} is an eigenvalue for f^{-1}.

1.23 Lecture 21 Feb 2014

Eigenvalues of a triangular matrix Let $A \in M_n$ be upper triangular. Now λI is also triangular. Hence $B = \lambda I - A$ is also triangular. Now the characteristic polynomial is the determinant of B. But we know that the determinant of a triangular matrix is just the product of the diagonal entries. Hence $\det(B) = \prod_{i=1}^{r}(x - a_{ii})$. Hence the eigenvalues are the entries of A along the main diagonal. The multiplicity of each eigenvalue is the number of times it appears on the main diagonal.

1.24 Lecture 24 Feb 2014

Complex conjugation Define $\sigma : \mathbb{C} \to \mathbb{C}$, $c \mapsto \bar{c}$. Recall that it preserves addition and multiplicaiton: $\sigma(c+d) = \sigma(c) + \sigma(d)$ and $\sigma(c \cdot d) = \sigma(c) \cdot \sigma(d)$.

Definition: Inner Product An inner product on vector space V is a bilinear function $B : V \times V \to F$ that satisfies three axioms:

1. B is hermitian symmetric: $\forall x, y \in V, B(y,x) = B(\bar{x}, y)$.
2. B is linear in its first variable: $\forall x, y, z \in V, \forall a, b \in F, B(ax + by, z) = aB(x,z) + bB(y,z)$.
3. B is positive definite: $\forall o \neq x \in V, o < B(x,x) \in \mathbb{R}$.

Write (x,y) for $B(x,y)$.

Remark 1 For $x \in V$, define $B_z : V \to F$ with $v \mapsto B(v,z)$. Observe that Axiom 2 is equivalent to requiring that $\forall z \in V$, B_z is linear.

Remark 2 If $\mathbb{F} = \mathbb{R}$, then $\bar{a} = a$, and the Hermitian symmetry is just ordinary symmetry $B(y,x) = B(x,y)$. Then along with Axiom 2, we have that B is also linear in the 2nd variable. $B(x, ay+bz) = B(ay+bz, x) = aB(y,x) + bB(z,x) = aB(x,y) + b(x,z)$. Hence B is bilinear. However, if $\mathbb{F} = \mathbb{C}$, then $B(z, ax+by) = \overline{B(ax+by, z)} = \overline{aB(x,z) + bB(y,z)} = \bar{a} \cdot \overline{B(x,z)} + \bar{b} \cdot \overline{B(y,z)} = \bar{a}B(z,x) + \bar{b}B(z,y)$. Hence in general, B preserves addition in the second variable $B(z, x+y) = B(z,x) + B(z,y)$ but it only preserves scalar multiplication up to a complex conjugation.

Example 1 Let $V = V_n(\mathbb{F})$. The dot product on V_n is the map $B(x,y) = x \cdot y = \sum_{k=1}^{n} x_k \overline{y_k}$, with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Example 2 Let $\mathbb{F} = \mathbb{R}$ and $[c,d]$ a closed interval on the reals. Take V to be a function space, the space of all real valued continuous functions on $[c,d]$. Define the inner product $B(f,g) = \int_c^d f(x)g(x)dx$.

Inner Product Space Let $(,)$ be an inner product on V. Call V together with an inner product an inner product space. Recall that for $z \in V$, $B_z : V \to F$ with $v \mapsto (v,z)$ is linear. Hence by the characterization of linear functions, we have that B_z maps the zero vector to the zero element $o = B_z(o) = (o,z)$. Also, as the inner product is positive definite, $(x,x) = 0$ iff $x = 0$. Also, if $(u,v) = 0$, then $0 = (u,v) = \overline{(v,u)} = \bar{0} = 0$. Hence $(u,v) = 0$ iff $(v,u) = 0$.

Norm function For $v \in V$, define the norm of v to be $||v|| \equiv \sqrt{(v,v)}$. In Euclidean space, $||v|| = \sqrt{\sum_{k=1}^{n} v_k \overline{v_k}} = \sqrt{\sum_{k=1}^{n} |v_k|^2}$.

Theorem 1.9 For $uv, \in V, a \in F$, (1) $||v|| \geq 0$ with equality iff $v = 0$. (2) $||aV|| = |a| \cdot ||v||$. (3) Triangle inequality: $||u+v|| \leq ||u|| + ||v||$.

Perpendicular For $x \in V$, define $x^{\perp} = \{v \in V : (v,x) = 0\}$. Read as x-perp. Say that v is orthogonal to x if $v \in x^{\perp}$.

1.25 26 Feb 2014 Lecture

Theorem 1L Let $u, v \neq 0$ be vectors in the plane. Let θ be the angle between v and u. Then $\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||}$.

Proof Write u,v in terms of Cartesian coordinates: $u = ||u||(\cos\phi, \sin\phi), v = ||v||(\cos(\theta+\phi), \sin(\theta+\phi))$, where ϕ is the angle between u and the x-axis. We multiply as per the definition of the dot product to obtain the statement.

Remark 4 The dot product measures the "angle" between vectors in a plane $V_2(\mathbb{R})$. u and v are orthogonal iff $u \cdot v = 0 \iff \cos\theta = 0 \iff \theta = \pi/2$ or $3\pi/2$, iff u and v are perpendicular.

Orthogonal set A subset S of V is orthogonal if for all distinct $u, v \in S$, $(u,v) = 0$.

Theorem 1.10 Let S be an orthogonal subset of nonzero vectors of V. Then the following are true: (1) S is independent, (2) If $|S| = \dim(V)$, then S is a basis for V.

Proof Let $\{s_1, \ldots, s_n\} \subseteq S$. As S is orthogonal $\forall i \neq j, (s_i, s_j) = 0$. Suppose $a_j \in \mathbb{F}, 1 \leq j \leq n$ such that $\sum_{j=1}^{n} a_j s_j = 0$. Then $\forall k, (0, s_k) = 0 = (\sum_j a_j s_j, s_k) = \sum_j a_j (s_j, s_k)$ by linearity in the first variable. But all the inner products are zero except for (s_k, s_k). Then $0 = a_k(s_k, s_k)$. But by hypothesis, $s_k \neq 0$, then $(s_k, s_k) \neq 0$. Then $a_k = 0$. Hence all the coefficients are zero, and the set S is independent. By Theorem 1K, if $\dim V = |S| \neq \infty$ and $S \subseteq V$ is independent, then S is a basis.

S-perp is a subspace Define $S^{\perp} = \{v \in V : (v, s) = 0 \forall s \in S\}$. So $S^{\perp} = \cap_{s \in S} s^{\perp}$. $S^{\perp}$ is a subsapce of S.

Proof Recall that the intersection of subspaces is a subspace. It suffices to show that $x^{\perp} \leq V$ for each $x \in V$. Recall that $B_x : V \to F$ is linear with $v \mapsto (v, x)$. Observe that the null space $N(B_x) = x^{\perp}$. But we know that the null space of a linear map is a subspace, hence $x^{\perp}$ is a subspace.

Direct Sum If $U, W \leq V$ then $U + W = \{u + w : u \in U, w \in W\}$ is a subspace of V. $U + W = \text{Ł}(U \cup W)$. We say that V is the direct sum of U and W if $V = U + W$ and $U \cap W = 0$. Write $V = U \bigoplus W$.

Theorem 1.15 Assume $\dim V = n \leq \infty$. Let $U \leq V$. Then (1) $U^{\perp}$ is a subspace of V and $\dim(U^{\perp}) = n - \dim U$. (2) $V = U \bigoplus U^{\perp}$ and (3) $(U^{\perp})^{\perp} = U$. (4) Each $v \in V$ can be written uniquely in the form $v = u + w$ with $u \in U$ and $w \in U^{\perp}$.

Proof For (1) and (2), Let $\{x_1, \ldots, x_m\}$ be a basis for U. Define $f_i : V \to F$ with $v \mapsto (v, x_i)$ so $f_i = B_{x_i}$. Then f_i is linear and $N(f_i) = x_i^{\perp}$. Notice that the dimension of the image is 1, since it is just the field F, so by Theorem 2,3 then $\dim(x_i^{\perp}) = \dim(N(f_i)) = \dim(V) - \dim(F) = n - 1$. Hence $codim(x_i^{\perp}) = 1$. Now as $U = L(x_1, \ldots, x_n)$ as X is a basis for U, then $U^{\perp} = \cap_{i=1}^{m} x_i^{\perp}$, left as Exercise.

Then by HW3Q2, $codim(U^{\perp}) \leq \sum_{i=1}^{m} codim(x_i^{\perp})$. But we know that $codim(x_i^{\perp}) = 1$. Hence $codim(U^{\perp}) \leq m$. Hence $dim(U^{\perp}) = n - codim(U^{\perp}) \geq n - m$. Now suppose $u \in U \cap U^{\perp}$. We know that $\forall v \in U^{\perp}, \forall u \in U, (v, u) = 0$. But this means that $(u, u) = 0$. But by definition, the inner product of a vector with itself is positive definite. Hence $U \cap U^{\perp} = 0$. Now apply the Intersection-Sum Dimension Theorem. Then $\dim(U + U^{\perp}) = \dim(U) + \dim(U^{\perp}) - \dim(U \cap U^{\perp}) \geq m + (n - m) - 0 = n$. But we know that the subspace of a vector space of dimension n is less or equal to n, with equality iff the subspace is the vector space itself. Hence $\dim(U + U^{\perp}) = n$ and $U + U^{\perp} = V$, so (2) holds. All inequalities are now equalities and $\dim(U^{\perp}) = n - m$ so (1) holds. To prove (3), we realize that each $u \in U$ is orthogonal of $U^{\perp}$ by definition of $U^{\perp}$ so $U \subseteq (U^{\perp})^{\perp}$. By Part (1), $\dim((U^{\perp})^{\perp}) = n - \dim(U^{\perp}) = n - (n - m) = m = \dim(U)$. Hence $U = (U^{\perp})^{\perp}$, so (3) holds. To prove (4), $\forall v \in V, \exists u \in U, w \in U^{\perp}$ with $v = u + w$. To show that this is unique, we suppose $v = u' + w'$ for some other $u' \in U$ and $w' \in U^{\perp}$. Hence $u + w = u' + w'$ and $u - u' = w' - w$. But since U and $U^{\perp}$ are subspaces, $u - u' \in U$ and $w' - w \in U^{\perp}$. But we know that $U \cap U^{\perp} = 0$, hence the only element that is in both U and $U^{\perp}$ is 0. Hence $u - u' = 0$ and $w' - w = 0$ and $u = u'$ and $w = w'$, hence the statement is unique, and (4) is true.

Lemma 5A If $U \leq V$, then the restriction of the inner product $(,)$ to U is an inner product space on U.

1.26 27 Feb 2014 Recitation

Positive definite A matrix A is positive definite if $x^*Ax \geq 0, \forall x \in C_n$ and equality iff $x = 0$. We realize that $x^*Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \bar{x}_i x_j$. If we define $(x, y) = y^*Ax$, then $(,)$ is an inner product on C_n. Realize that $(y, x) = (y^*Ax) = (y^*Ax)^t = (y^*Ax)^* = x^*A^*(y*)^* = x^*Ay = (x, y)$.

Orthogonal A matrix $A \in M_n(\mathbb{R})$ is orthogonal iff $AA^t = I$. Write $AA^t = (b_{ij}) = \sum_{k=1}^{n} a_{ik}a_{jk}$. For (b_{ij}) to be the identity, we require that $\sum_{k=1}^{n} a_{ik}^2 = 1$ and $\sum_{k=1}^{n} a_{ik}a_{jk} = 0$. Let r_j be the jth row of A. Then we require that $(r_j, r_j) = 1$ and $(r_i, r_j) = 0$ for all i,j. Hence A is orthogonal iff the rows of A form an orthonormal set (note that this has to be orthonormal since the norm of each row is 1). Similarly, using $A^TA = I$ from the definition of the inverse, we see that the columns are orthonormal too.

Unitary matrix A matrix $A \in M_n(F)$ is unitary iff $AA^* = I$. Similarly, the rows of A form an orthonormal set and the columns of A also form an orthonormal set. Note that the inner product used here is $(x, y) = \sum_i x_i \bar{y_i}$ with complex conjugation on the second variable coordinates.

1.27 28 Feb 2014 Lecture

Lemma 5B Let $0 \neq x \in V$. Then $\exists y \in Fx$ a scalar multiple of x with $(y, y) = 1$.

Proof Since $x \neq 0$, by the positive definite axiom, we know that $(x, x) > 0$. Hence $\exists 0 \neq a \in \mathbb{R}$ with $a^2 = (x, x)$. Then we can form $y = a^{-1}x$ such that $(y, y) = (a^{-1}x, a^{-1}x) = a^{-1}a^{-1}(x, x)$ by linearity, no need for complex conjugate since a is real. But is equal to $(x, x)/(x, x) = 1$.

Orthonormal basis An orthonormal basis over V is a basis X such that $\forall x, y \in X, (x, y) = \delta_{x,y}$.

Theorem 1.14 If $0 < \dim(V) < \infty$, then V has an orthonormal basis.

Proof Let $0 \neq x \in V$. Let n be the dimension of the space V. By Lemma 5B, $\exists x_1 \in Fx, (x_1, x_1) = 1$. If $n = 1$, then we are done, since $\{x_1\}$ is an orthonormal basis. So we may take n=1, and proceeding by induction, we assume that the theorem holds for spaces of dimension $n-1$. Let U be the one-dimensional vector space generated by x_1. Write $W = U^{\perp}$. By Theorem 1.15, $\dim W = n - 1$. By Lemma 5A, the inner product is also an inner product in W. So by the induction assumption, $\exists$ an orthonormal basis $\{x_2, \ldots, x_n\}$ for W. Let $X = \{x_1, x_2, \ldots, x_n\}$. We observe that X is an orthonormal basis for V.

Maps between inner product spaces Let V' be an F-space with inner product $(,)'$. A linear map $f : V \to V'$ is unitary if f preserves the inner product. This means that $\forall x, y \in V, (x, y) = (f(x), f(y))'$.

Isometry An isometry if $V, (,)$ with $V', (,)'$ is a vector space isomorphism $f : V \to V'$ which is unitary. Note that if $f : V \to V'$ is an isomorphism, then $f^{-1} : V' \to V$ is also an isometry. Note that to prove this, we just need to check that f^{-1} preserves inner products, since f is an isomorphism, so we know that the inverse exists and the inverse is an isomorphism also. We say that V and V' are isometric if $\exists$ an isometry between them. Geometrically, unitary maps preserve distance and angle.

Example of isometry Consider $V = \mathbb{R}^2$ under the dot product. Define $f : V \to V$ by $f(v_1, v_2) = (v_2, -v_1)$. This is effectively a rotation through $-\pi/2$.

Theorem 5.17 Let $f : V \to V'$ be an isomorphism. Let X be a basis for V. Then f is an isometry iff $\forall x, y \in X, (x, y) = (f(x), f(y))'$.

Proof Forward direction is trivial. For the reverse direction, we let $u, v \in V$. Then $(u, v) = (\sum_i a_i x_i, \sum_j b_j x_j) = \sum_{i,j} a_i \bar{b_j} (x_i, x_j) = \sum_{i,j} a_i \bar{b_j} (f(x_i), f(x_j))$ by assumption. This is equal to $(\sum_i a_i f(x_i), \sum_j b_j f(x_j))' = (f(u), f(v))'$.

Theorem 5C If $\dim V = n < \infty$, then V is isometric to V_n under the dot product. That is, V is an n-dimensional Euclidean space over F.

Proof By Theorem 1.14, V has an orthonormal basis X. We also know that the set of coordinate vectors B is orthonormal. Hence we just need to find a linear map to map basis X to B. By Theorem 2.12, there exists a linear map $f : V \to V_n$ such that $f(x_i) = e_i, \forall i$. By Theorem 2C, $f : V \to V_n$ is an isomorphism. It remains to show that f is unitary. But $(x_i, x_j) = \delta_{ij} = e_i \cdot e_j = f(x_i) \cdot f(x_j)$. So by Theorem 5.17, f is unitary.

Theorem 5.15 If $\dim V < \infty$, then each unitary map $f : V \to V$ is an isometry.

Proof We must show that f is an isomorphism. By Theorem 2C, we just need to check that its null space is zero. Let $x \in N(f)$ .Then $f(x) = 0$, so $0 = (0, 0) = (f(x), f(x)) = (x, x) \implies x = 0 \implies N(f) = 0$.

Theorem 5.16 Assume $\dim V = n < \infty$, and $\mathbb{F} = \mathbb{C}$. Let $f \in \mathcal{L}(V)$ be unitary. Then (1) there exists an orthonormal basis X of eigenvectors for f, (2) $|a| = 1$ for all eigenvalues a of f, (3) $m_X(f)$ is $diag(a_1, a_2, \ldots a_n)$, where a_i are the eigenvalues of f.

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Theorem 5.16 Assume $\dim V = n < \infty$ and $F = \mathbb{C}$. Let $f \in \mathcal{L}(V)$ be unitary. Then (1) there exists an orthonormal basis X of eigenvectors for f. (2) $|a| = 1, \forall$ eigenvalues a of f (3) $m_X(f) = diag(a_1, \ldots, a_n)$, where a_i are the eigenvalues of f.

Proof Let $p(x) = char_f(x)$. Then $p(x) = (x - a_1) \ldots (x - a_n)$. As $F = \mathbb{C}$, by Theorem 4.5, $a_1, \ldots, a_n$ are the eigenvalues of f. So there exists an eigenvector x_1 for a_1. By Theorem 5B, there exists $x_1' \in Fx_1$ such that $(x_1', x_1') = 1$. Now as x_1' is also an eigenvector for a_1, we replace x_1 by x_1', and we have $(x_1, x_1) = 1$. By Theorem 5.15, any unitary map f is an isomorphism of V with V. Hence, the null space of f is zero. But the null space of f is the zero eigenspace. Hence we conclude that the zero eigenspace only includes the zero vector, hence there are no non-zero eigenvectors for the zero eigenvalue. Hence zero is not an eigenvalue, and all eigenvalues are non-zero. Now let U be the space of all scalar multiples of x_1 and let $W = U^{\perp}$. Then by Theorem 1.15, $V = U \bigoplus W$. Also $\dim(W) = n - \dim(U) = n - 1$. We claim that $f(W) \subseteq W$. To prove the claim, we let $w \in W$ and we require $f(w) \in W$. Now we let $(x_1, w) = 0 = (f(x_1), f(w))$ since f is unitary, so it preserves the inner product. This is equal to $(a_1 x_1, f(w)) = a_1(x_1, f(w))$. Now we know that $a_1 \neq 0$, hence $(x_1, f(w)) = 0$, and $f(w) \in x_1^{\perp} = W$. Now by Theorem 5A, W (with (,)) is an inner product space. We perform induction over the dimension n. We note that the statement is true for $n = 1$. Let it be true for $n-1$ dimensional spaces. By the claim, and noting that f is unitary on V, $f \in \mathcal{L}(W)$ and f is unitary on W. Hence there exists an orthonormal basis $\{x_2, \dots, x_{n-1}\}$ of eigenvectors for f on W. Now let X be the set of eigenvectors with x_1 included. Hence X is orthonormal. Now since the set is orthogonal, it is also independent, and since it is of the right order n, it is a basis for V. This proves (1).

To prove (2), we examine the inner product of an eigenvector with the orthonormal eigenvector basis $(x_i, x_i) = 1 = (f(x_i), f(x_i)) = (a_i x_i, a_i x_i) = a_i \bar{a}_i (x_i, x_i) = |a_i|^2 = 1$. Hence $|a_i| = 1$.

To prove (3), as X is a basis of eigenvectors for linear map f, by Lemma 4B, the matrix of f with respect to X is diagonal, with the diagonal entries being the eigenvalues of f.

Unitary Matrix Let $A = (a_{ij}) \in M_n(\mathbb{F})$. A is unitary if $AA^* = I$.

Remark 5 A is unitary iff A is invertible and $A^* = A^{-1}$. Hence A^* is also unitary, and A is unitary iff $A^*A = I$.

Proof If A is unitary, then by definition $AA^* = 1$. We know that $\det(I) = 1$. Hence $\det(A)\det(A^*) = 1$. Hence the determinant of A is non-zero, hence it has an inverse by Lemma 3H. Also, $A^{-1} = A^{-1}I = A^{-1}AA^* = (A^{-1}A)A^* = IA^* = A^*$. For the other direction, if A is invertible with $A^{-1} = A^*$, then $AA^* = AA^{-1} = I = A^{-1}A = A^*A$. Hence if A is unitary and $A^*A = I$.

Remark 6 $(AB)^* = B^*A^*$ and $(AB)^t = B^tA^t$.

Proof Let $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$. Then $A^t = (a_{ji}), B^t = (b_{ji})$. Let $B^tA^t = (d_{ij})$. $d_{ij} = \sum_k b_{ki}a_{jk} = \sum_k a_{jk}b_{ki} = c_{ji}$. Hence the statement is true.

Remark 7 A matrix (a_{ij}) is real if all its entries are real. If A is real then $A^* = A^t$. Hence when A is real, A is unitary iff $AA^t = I$.

Lemma 5D Let $X = \{x_1, \dots, x_n\}$ be a basis for V. Define the matrix of $(,)$ with respect to the basis to be $B = (b_{ij)} \in M_n(\mathbb{F})$ such that $b_{ij} = (x_i, x_j)$. Let $f \in \mathcal{L}(V)$ and $A = m_X(f)$. Then f is a unitary linear map iff $B^t = A^*B^tA$.

Proof By Theorem 5.17, a map f is unitary iff $(x_i, x_j) = (f(x_i), f(x_j)), \forall i, j$. We note that this is b_{ij}. Let $A = (a_{ij})$. Then $(f(x_i), f(x_j)) = (\sum_k a_{ki}x_k, \sum_l a_{lj}x_l) = \sum_{k,l} a_{ki}\bar{a_{lj}}(x_k, x_l) = \sum_{k,l} a_{ki}\bar{a_{lj}}b_{kl} = \sum_{kl} a^t_{ik}b_{kl}\bar{a_{lj}} = i, j$th entry in $A^tB\bar{A}$. Hence, f is unitary iff $B = A^tB\bar{A}$. Taking the transpose of the equation, $B^t = (A^tB\bar{A})^t = \bar{A}^tB^tA = A^*B^tA$. Hence the statement is true.

Theorem 5.18 Let X be an orthonormal basis of V and $f \in \mathcal{L}(V)$. Then f is unitary iff $m_X(f)$ is unitary.

Proof As X is orthonormal, the matrix B of $(,)$ with respect to X is I. $B = (b_{ij}) = ((x_i, x_j)) = (\delta_{ij}) = I$. Now let $A = m_X(f)$. By Lemma 5D, f is unitary iff $B^t = A^*B^tA$. But since B is the identity, we require that $I = A^*A$, which is equivalent to saying that A is unitary.

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Theorem 5.19 If A is unitary, then (1) A is similar over $\mathbb{C}$ to a diagonal matrix (2) the eigenvalues of A have modulus 1.

Proof Let X be an orthonormal basis of V, $\dim V = n$. Then $A = m_X(f)$ for some $f \in \mathcal{L}(V)$. As A is unitary, f is also unitary by Theorem 5.18. By Theorem 5.16, the unitary map has a basis Y of eigenvectors. Then $m_Y(f)$ is diagonal, with the entries along the main diagonal equal to the eigenvalues of f. Also, by Theorem 5.16, these eigenvalues have absolute value 1. Now by Theorem 4.8, $m_Y(f)$ is similar to $A = m_X(f)$ over $\mathbb{C}$. Hence these exists some invertible matrix $C \in M_n(\mathbb{C})$ such that $C^{-1}AC = m_Y(f)$. Also, $m_Y(f)$ has the same eigenvalues as $m_X(f)$.

Hermitian Map $f \in \mathcal{L}(V)$ is Hermitian if $\forall x, y \in V$, $(f(x), y) = (x, f(y))$.

Hermitian Matrix $A \in M_n$ is Hermitian if $A = A^*$, A is self-adjoint. Hence A is Hermitian iff $a_{ij} = \bar{a_{ji}}, \forall i, j$.

Symmetric matrix We say $A \in M_n$ is symmetric iff $a_{ij} = a_{ji}, \forall i, j$. A is symmetric around the diagonal.

Notice Recall A is real if all its entries are real. If A is real, then $\forall i, j, \bar{a_{ij}} = a_{ij}$. Hence a real matrix A is hermitian iff A is symmetric.

Theorem 5.4 Let $\dim V = n < \infty$ and assume $f \in \mathcal{L}(V)$ is hermitian. Then, (1) there exists an orthonormal basis X of eigenvectors for f, (2) the eigenvalues of f are real, (3) $m_X(f)$ is diagonal.

Proof Refer to proof of Theorem 5.16(1) for proof of (1) and (3). To prove (2), let a be an eigenvalue of F. Then there exists an associated eigenvector x for a. By Lemma 5B, we can pick x such that $(x, x) = 1$. Write $a = a(x, x) = (ax, x) = (f(x), x) = (x, f(x)) = (x, ax) = \bar{a}(x, x) = \bar{a}$. Hence $a = \bar{a}$, hence a has to be real.

Theorem 5.6 Let $X = \{x_1, \ldots, x_n\}$ be an orthonormal basis for V. Let $f \in \mathcal{L}(V)$ and $A = m_X(f)$. Then f is hermitian iff $m_X(f)$ is hermitian.

Proof We note that f is hermitian iff $\forall i, j, (x_i, f(x_j)) = (f(x_i), x_j)$. $LHS = (x_i, f(x_j)) = (x_i, \sum_k a_{kj}x_k) = \sum_k \overline{a_{kj}}(x_i, x_k) = \sum_k \overline{a_{kj}}\delta_{ik} = \overline{a_{ij}}$. Similarly, $(f(x_i), x_j) = a_{ji})$. Hence f is hermitian iff $\forall i, j \overline{a_{ij}} = a_{ji}$, hence A is hermitian. Hence f is hermitian iff A is hermitian.

Theorem 5.7 Let $A \in M_n(\mathbb{F})$ be hermitian. Then (1) A is similar over F to a diagonal matrix, and (2) the eigenvalues of A are real.

Proof Pick an orthonormal basis X for V. Then $A = m_X(f)$ for some $f \in \mathcal{L}(V)$. As A is unitary, f is also unitary by Theorem 5.6. Hence by Theorem 5.4(3), there exists a basis Y for V such that $m_Y(f)$ is diagonal, with eigenvalues along the main diagonal. Also, by Theorem 5.4(2), the eigenvalues of f are real. Now by Theorem 4.8, $m_Y(f)$ is similar to $m_X(f)$ over $\mathbb{F}$. Hence there exists some invertible matrix $C \in M_n(\mathbb{F})$ such that $C^{-1}AC = m_Y(f)$. Hence (1) holds. Also, $m_Y(f)$ has the same eigenvalues as $m_X(f)$, hence the eigenvalues of $m_X(f)$ are real also. Hence (2) holds.

Corollary Let A be a real symmetric matrix. Then (1) A is similar over $\mathbb{R}$ to a diagonal matrix (2) the eigenvalues of A are real.

Proof Since $A \in M_n(\mathbb{R})$, hence we may take $\mathbb{F} = \mathbb{R}$. Since A is real symmetric, A is also hermitian. Hence we appeal to Theorem 5.7 to conclude that (1) A is similar to a diagonal matrix and (2) the eigenvalues of A are real.

Cayley-Hamilton Theorem Theorem 7.8 in the text. Let $A \in M_n(\mathbb{C})$ and $p(x) = char_A(x)$ Then $p(A) = 0$. Recall that a matrix polynomial $q(A) = \sum_{i=0}^{m} c_i A^i$ where $A^0 = I$ and A^i is the ith power of A.

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Direct Sums of Matrices Let $0 < n \in \mathbb{Z}$. Let $C \in M_n$ and let $0 < m < n$. Let $A \in M_m$ and $B \in M_{n-m}$. Write $C = A \bigoplus B$, the direct sum of A and B. This indicates that C has the form: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Recall that $\det(C) = \det(A)\det(B)$. More generally, suppose $A_1, A_2, \ldots A_r$ are square matrices with $A_i \in M_{n_i}$. Let $n = n_1 + n_2 + \ldots + n_r$. Write $C = A_1 \bigoplus \ldots \bigoplus A_r$ if C has the matrices A_i along the main diagonal (all other entries zero). Note that $C \in M_n$. This is called a block diagonal decomposition.

Jordan Block Define $N_n \in M_n$ by $N_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. Then $N_n = (b_{ij})$ where $b_{ij} = \begin{cases} 0 & \text{if } j \neq i+1 \\ 1 & \text{if } j = i+1 \end{cases}$. Then N_n has 1s on the super diagonal and zeroes elsewhere. A matrix $A \in M_n$ is a Jordan block if size n if $A = aI + N_n$ for some $a \in \mathbb{F}$. Then A has values of a along the main diagonal and 1s along the super diagonal. In particular, A is triangular.

Jordan Form A matrix $A \in M_n$ is in Jordan form if for some positive integer r and for $1 \leq i \leq r$, there exists positive integers $n_1, \ldots, n_r$ such that $n = n_1 + \ldots + n_r$ and there exists Jordan blocks $A_1, \ldots A_r$ with A_i of size n_i such that $A = A_1 \bigoplus \ldots \bigoplus A_r$.

Jordan Form Theorem Each matrix $A \in M_n(\mathbb{C}$ is similar over the complex numbers to a unique (up to ordering of Jordan blocks) matrix in Jordan form.

Example Take $n = 2$. $A \in M_2$ is in Jordan form iff either (1) A is the direct sum of two Jordan blocks of size 1, (2) A is a Jordan block of size 2. In case 1, $A_i = a_i I$ for some scalar $a_i \in \mathbb{C}$. Hence $A = A_1 \bigoplus A_2 = diag(a_1, a_2)$. In case 2, $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

Remark Let $\dim V = n$ over the complex numbers. Also let $f \in \mathcal{L}(V)$ and X a basis for V. Let $A = m_X(f)$. By the Jordan Form theorem, there exists some matrix A' in Jordan form similar to A, so there exists some basis X' with $m_{X'}(f) = A'$. Suppose $A = A'$. Say $A = A_1 \bigoplus \dots \bigoplus A_r$ with $A_i = a_i I + N_{n_i}$. Then (1) matrix A is upper triangular, hence its eigenvalues are the entries along the diagonal, hence are $a_1, \dots a_r$. Define $J(a_i) = \{j : a_i = a_j\}$. Then (2) the multiplicity of a_i as an eigenvalue of f is $\sum_{j \in J(a_i)} n_j$. Also, (3) for $a \in \{a_1, \dots, a_r\}$, $\dim(E(a)) = |J(a)|$. Also, (4) A is similar to a diagonal matrix iff all the Jordan blocks of A are of size 1. If any block is of size greater than 1, then A is not similar to a diagonal matrix.

Proof For (1) and (2), as A is the direct sum of upper triangular matrices, it is also upper triangular. We know that in a triangular matrix, the eigenvalues are the entries along the diagonal. (3) as an exercise (4) Assume A is in Jordan Form. Then A is diagonal iff each Jordan block is diagonal. But each Jordan block is diagonal only when it is of size 1. Suppose A is similar to a diagonal matrix D. Then D is in Jordan form, since D has n Jordan blocks of size 1. But we know that A is similar to a unique matrix in Jordan form, so D is the unique (subject to re-ordering of Jordan blocks) Jordan form of A. So $A = D$.

Example Let V be the space of real polynomials of degree less than n for some n. Then $Y = \{1, x, x^2, \dots x^{n-1}\}$ is a standard basis for V. Let $f'(x) = \sum_{i=0}^{n-1} i a_i x^{i-1} \in V$. Define $\delta : V \to V$ with $f \mapsto f'$. This is a linear map. Define $x_i = x^i / i!$ where $0! \equiv 1$. Then $X = \{x_i : 0 \leq i < n\}$ is also a basis for V. Then $\delta(x_0) = 0$, and for some $i > 0$, then $\delta(x_i) = (x^i/i!)' = i x^{i-1}/i! = x^{i-1}/(i-1)! = x_{i-1}$. Then δ maps each basis vector to the previous basis vector in the ordered basis X. Then $m_X(\delta) = N_n$, with 1s along the super diagonal and zeroes elsewhere. Then $m_X(\delta)$ is a Jordan block of size n.