ACM95a Class Notes

Lim Soon Wei Daniel

Chapter 1

Week 1

1.1 Monday, 29 Sept 2014

Intro Dale Pullin, x6081, Gugg 306F, dpullin@caltech.edu

TA: TBA

No recitation this week!

Review of complex numbers: Let $a, b \in \mathbb{R}$. Write $\alpha = a + ib$ or (a, b).

Properties of i: $i^2 = -1, i^3 = -i, i^4 = 1$ and so on.

Laws obeyed by complex numbers: Commutative, Associative, Distributive.

Features of complex numbers: Complete (Fundamental Theorem of algebra), Efficient method of calculating integrals, physically relevant (QM)

Complex conjugate Is its own inverse, distributive, and $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

Modulus: $|\alpha| \equiv \sqrt{a^2 + b^2}, |\alpha\beta| = |\alpha||\beta|$. Triangular inequality: $|\alpha + \beta| \leq |\alpha| + |\beta|$.

1.2 Wednesday, 1 Oct 2014

Square Root Consider $w = \zeta + i\eta, z = x + iy$, with $w = z^{1/2}$. Then $\eta = \frac{y}{2\zeta}, \zeta = \pm\sqrt{\frac{1}{2}(x + \sqrt{x^2 + y^2})}$. Difficult and cannot be extrapolated to higher roots.

Argument function Angle of the "vector" in the Argand diagram made with the positive x-axis, measured anti-clockwise.

Modulus-Argument Form: $x + it = r(\cos\theta + i\sin\theta)$.

Multivaluedness: arg is a multivalued function. To make θ unique, we pick $\theta = \text{Tan}^{-1}(x, y)$, which will satisfy the following properties: 1. $\tan[\text{Tan}^{-1}(x, y)] = \frac{y}{x}$, and 2. $-\pi < \text{Tan}^{-1}(x, y) \leq \pi$, and 3. $\sin[\text{Tan}^{-1}(x, y)] = \frac{y}{\sqrt{x^2+y^2}}, \cos[\text{Tan}^{-1}(x, y)] = \frac{x}{\sqrt{x^2+y^2}}$. Call $\theta = \text{Arg}(z)$, the principal branch of the multivalued function $\theta = \arg z$. It has a discontinuity along the negative real axis.

Multiplication in polar form Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$. Hence $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \mathbb{Z}$. Note that $\text{Arg}(z_1 z_2) \neq \text{Arg } z_1 + \text{Arg } z_2$. $\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2k'\pi$ for a particular k'.

de Moirve's theorem $z^n = r^n(cos(n\theta) + i\sin(n\theta))$. In particular, for $n = 1$, $z^0 = 1$. It can also be shown that $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$ and that $\arg\left(\frac{1}{z}\right) = -\arg(z) + 2k\pi$, $\arg(-z) = (\arg z - \pi) + 2k\pi$. Also works for $n < 0, n \in \mathbb{Z}$.

Integer Roots Let $n \in \mathbb{Z}^+$. Let $w = z^{1/n}$ such that $w^n = z$. Let $w = \rho(\cos\phi + i\sin\phi)$. Then $w^n = \rho^n(\cos(n\phi) + i\sin(n\phi)) = z = r(\cos\theta + i\sin\theta)$. Hence $\rho = \sqrt[n]{r}$. Note that $\phi = \frac{\theta}{n}, \frac{\theta+2\pi}{n}, \ldots, \frac{\theta+2\pi(n-1)}{n} = \frac{\theta+2k\pi}{n}, k = 0, 1, \ldots, n-1$.

Terminology A **curve** C or ∂D is a set of points defined by continuous functions $x(t), y(t)$ on $a \le t \le b$. An **open set** is a set of points $\hat{R}$ of the plane such that every point is an interior point. We say that the point z_0 is in $\hat{R}$ if there exists an ϵ such that $z \in \hat{R}$ whenever $|z - z_0| \le \epsilon$. An **open connected set** is a set of points such that any two points in the set can be connected by a polygonal path. A **domain** is an open connected set. A domain is **bounded** if there exists M such that $|z| < M$ for each point in the domain. A **region** is a domain together with some or none of the boundary points. A **closed region** contains all its boundary points. A **bounded/finite region** is such that there exists M such that $|z| \le M$ for all z in the region. A **compact region** is a closed, bounded region.

Theorem Let $u(x, y)$ be real valued in a domain D. If $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$ at all points in D, then u is constant in D.

Key Hole Domain/Region Excludes the origin and the negative real axis. Interior to a circle.

1.3 Friday, 03 Oct 2014

Mappings and Functions If $\forall z \in \{D, R\}$ there exists a rule which assigns a complex number w to z then $w = f(z)$ on $\{D, R\}$ (where D=Domain, R=Region). Write $w = u + iv$ and $z = x + iy$. Then we have the functions $u(x, y)$ and $v(x, y)$. Then the domain D or region R maps to an image in the uv plane $\mathcal{D}$ or $\mathcal{R}$. Say that $f(z)$ maps $\{D, R\}$ in the z-plane to $\{\mathcal{D}, \mathcal{R}\}$ in the uv-plane.

1-1 mapping Call a function/mapping one-one on some domain D iff: $f(z_1) = f(z_2) \implies z_1 = z_2$

Point at infinity Consider $w = f(z) = \frac{z}{1-z}$. Manipulate $\frac{z_1}{1-z_1} = \frac{z_2}{1-z_2}$ to obtain $z_1 = z_2$ except at $z_1 = 1 = z_2$. In the complex plane, surround the point $z = 1$ with a circle of radius ϵ. Then we have the curve $z = 1 + \epsilon e^{i\phi} = 1 + \epsilon(\cos\phi + i\sin\phi)$ to describe the circle. The image of the circle under the function w is $\frac{1+\epsilon e^{i\phi}}{-\epsilon e^{i\phi}}$. After some manipulation, $w + 1 = \frac{-1}{\epsilon}[\cos\phi - i\sin\phi]$. Hence as $\epsilon \to \infty$, the distance of w to the origin becomes infinite. Consider $w(\epsilon \to \infty)$ the point at infinity.

Bilinear or Mobius Mapping $w = \frac{z}{1-z}$ is a special case of the Bilinear or Mobius mapping. The general form is $w = \frac{az+b}{cz+d}$, with inverse $z = \frac{dW-b}{-cw+a}$.

Conformal Map Angle-preserving mapping except at singular points. Angle refers to the angle between two straight lines.

Notation Write $f(z) = u(x, y) + iv(x, y)$. The conjugate is $\overline{f(z)} = u(x, y) - iv(x, y)$. This is different from the function of the conjugate variable $\bar{z}$, which is $f(\bar{z}) = u(x, -y) + iv(x, -y)$. Extending this further, $\overline{f(\bar{z})} = u(x, -y) - iv(x, -y) = \bar{f}(z)$ and call this the conjugate function. For example, let $f = \alpha z^n$, so $\overline{f(z)} = \bar{\alpha}(\bar{z})^n$ and $\bar{f}(z) = \bar{\alpha} z^n$. Note that if $f(z)$ is analytic, then $\bar{f}(z)$ is also analytic, while $f(\bar{z})$ and $\overline{f(z)}$ may not be analytic.

Elementary Functions of a complex variable We have already seen algebraic functions, such as polynomials, rational functions, rational fractional powers.

Exponentials We define the exponential in such a way that it behaves in the same manner as the real counterpart e^x: $e^{z_1}e^{z_2} = e^{z_1+z_2}, e^{nz} = (e^z)^n$. Define $e^z = e^x(\cos y + i\sin y)$ and $z = re^{i\theta}$. Some properties: $\frac{d}{dz}e^z = e^z$. Note that $e^{z/n} \ne (e^z)^{1/n}$. Instead, $e^{z/n} = e^{x/n}\left[\cos\left(\frac{y}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{y}{n} + \frac{2\pi k}{n}\right)\right]$ for $k = 0, 1, \ldots, n-1$. We say that there are n branches of the multivalued function $(e^z)^{1/n}$.

Branch We say that a function $F(z)$ is a branch of the multivalued function $f(z)$ in D if $F(z)$ is continuous, single-valued, and for each $z \in D, F(z)$ is one of the values of $f(z)$. For example, let $f(z) = (e^z)^{1/5}$. There are 5 branches in this function. We can define a branch to be $F(z) = \left[(e^z)^{1/5}\right]\Big|_{k=3}$ by choosing the value of k.

Equality We say that two multivalued complex functions $f_1(z)$ and $f_2(z)$ are equal if there exists branches which are equal. For example, if $f_1(z) = e^{z/5}$ and $f_2(z) = [(e^z)^{1/5}]_k = 0$, then there are branches that are equal, so $f_1(z) = f_2(z)$.

Complete Equality We say that two multivalued complex functions $f_1(z)$ and $f_2(z)$ are completely equal when every branch of one function is equal to some branch of the other function.

Chapter 2

Week 2

2.1 Monday 6 Oct 2014

Complex exponential Recall that $e^z = 0$ has no finite roots, $e^z = 1 \implies \begin{cases} e^x = 1 \implies x = 0 \\ y = 2k\pi, k \in \mathbb{Z} \end{cases}$. Also, $e^{z_1} = e^{z_2} \implies z_1 = z_2 + 2\pi ik, k \in \mathbb{Z}$, hence the exponential mapping is not one-to-one. Let $w = e^z$ be a mapping. Write $w = u + iv = e^{x+iy} = e^x(\cos y + i \sin y)$ so $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

Complex exponential as a mapping Consider a line in the $x - y$ plane $z = x + ic$ where $-\infty < x < \infty, -\pi < c < \pi$. Then $w = e^z = e^x[\cos c + i \sin c]$ is just a radial line that goes from the origin to infinity in the uv plane. If we let $c = \pi - i\epsilon$, where $\epsilon << 1$, then $w = e^x(-1 + i\epsilon + O(\epsilon^2))$ is the line just above the negative real axis in the uv plane. Similarly, if $c = -\pi + i\epsilon$, then $w = e^x(-1 - i\epsilon)$ is a line just below the negative real axis in the uv plane.

Cuts Consider $D_0 \equiv \{-\pi < y < \pi\}$. Then e^z maps D_0 to $\mathcal{D}$, where $\mathcal{D}$ is the whole w plane excluding $u < 0, v = 0$. We note that we cannot cross the cut in the negative real axis in $\mathcal{D}$. Under these conditions, $w = e^z$ maps $D_0 \to \mathcal{D}$ in a 1-1 manner, and is continuous. Then $\mathcal{D}$ is the image of D_0 under the mapping $w = e^z$. We note that we could also have taken the strip $D_1 \equiv \{\pi < y < 3\pi\}$, which also maps to $\mathcal{D}$. There are an infinite number of strips that maps to $\mathcal{D}$. We can say $w = e^z$ is single-valued, but the inverse is not single-valued.

Trigonometric functions Recall that $e^z = e^x(\cos y + i \sin y)$. Along the imaginary axis, $z = iy, x = 0$, so $e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$. Rearranging, $\cos y = \frac{e^{iy}+e^{-iy}}{2}$ and $\sin y = \frac{e^{iy}-e^{-iy}}{2i}$. We define $\cos(z)$ and $\sin(z)$ accordingly. We can also write $e^{\pm iz} = e^{\mp y}(\cos x \pm i \sin x) = \cos z \pm i \sin z$. Also $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ and $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$ as expected for the real case. Further properties: $\cos(z) = \cos(-z)$ and $\sin(-z) = -\sin(z)$, $\cos(0) = 1, \sin(0) = 0, \cos^2 z + \sin^2 z = 1$.

Hyperbolic trigonometric functions $\cosh z \equiv \frac{e^z+e^{-z}}{2}, \sinh z \equiv \frac{e^z-e^{-z}}{2}$. It also can be shown that $\cos(iz) = \cosh(z), \sin(iz) = i \sinh(z)$ and $|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$, and that $\cos(z) = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$. As $y \to \pm\infty$, $|\cos z| \to \infty$. Similarly, $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$. When $y \to \pm\infty, |\sin z| \to \infty$. Also note that $\cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$. When $y \to \pm\infty$, $\cosh z$ becomes oscillatory. Also, $\sinh z = \sinh x \cos y + i \cosh x \sin y$.

Zeroes of trigo functions $\sin z = 0 \implies x = n\pi$ (so that $\sin x = 0$) and $\sinh y = 0$ (since $\sin x = 0$ so $\cos x \neq 0$) so $y = 0$. So the only roots of $\sin z = 0$ is $z = n\pi$, all lying on the real axis. Similarly, if $\sinh z = 0$ then $z = n\pi i$.

Complex logarithm Consider $w = \log z$ (note small l). Recall the definition of $\ln x = e^{\ln x} = x$, with $\ln(\infty) = \infty, \ln(1) = 0, \ln(0) = -\infty, \ln(e) = 1$. Also note that $\lim_{x\to\infty} [x^{-p} \ln x] \to 0$ for all $p > 0$ so it goes to infinity slower than any polynomial. We are going to require that $\log(x + i0) = \ln x$. Define $w = \log z$ as the root of $e^w = z$. We now can use the previous results about the exponential mapping by exchanging w and z. Hence $w = \log(z)$ exists for $0 < |z| < \infty$ since e^w maps the w plane for $-\pi < v < \pi$ into the whole z plane. But there exist infinitely many values of w corresponding to each value of z. Hence there are infinitely many regions of the w plane that map to the same region in the z-plane. Hence $w = \log z$ is multivalued.

2.2 Wednesday 8 Oct 2014

Multivalued Complex Logarithm Recall that $W = \log z$ is multivalued. Write $W = u + iv, z = x + iy$. Hence $e^w = e^u e^{iv} = z = re^{i\theta}$. Hence we have that $e^u = r$, so $u = \ln r$, the natural log. Also, $e^{iv} = e^{i\theta}$ so $v = \theta + 2\pi k, k \in \mathbb{Z}$. Hence $W = \log z = \ln r + i(\theta + 2\pi k), k \in \mathbb{Z} = \ln|z| + i \arg(z)$ since we have defined $\arg(z) = \theta + 2\pi k$ to be the multivalued argument function.

Aside Recall that $z = re^{i\theta} = |z|e^{i\arg(z)}$ so we can write $f = f(z) = |f|e^{i\arg(f)}$ and $\log f(z) = \ln|f| + i\arg f(z)$. Hence $\log f(z)$ is multivalued or single-valued depending on if $\arg f$ is multivalued or single-valued.

Example 1 Take $f = e^z = e^x(\cos y + i\sin y)$. Hence $|f| = e^x$ and $\arg(f) = y + 2\pi k$. Also, $\log e^z = x + i(y + 2\pi k) = (x + iy) + 2\pi ik = z + 2\pi ik$.

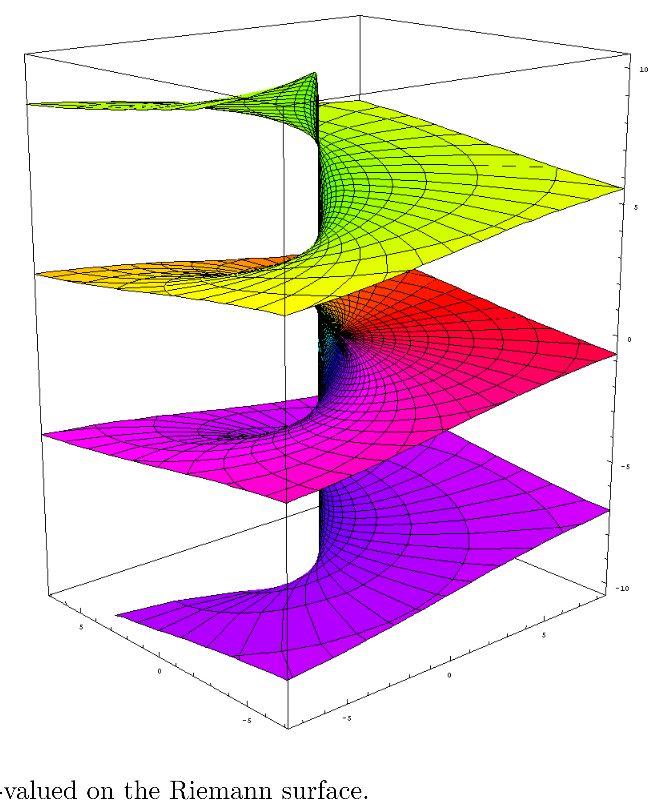
Example 2 Take $f(z) = e^{\log z} = e^{\ln|z| + i\arg(z)} = re^{i(\theta+2\pi k)} = re^{i\theta} = z$. Hence this function is single-valued.

Multiplication and Logarithms Recall that $\ln(x_1x_2) = \ln x_1 + \ln x_2$ and $\ln(1/x) = -\ln x$, since $x \equiv e^{\ln x}$ and $e^a e^b \equiv e^{a+b}$. Now consider $\log(z_1z_2) = \ln|z_1z_2| + i\arg(z_1z_2) = \ln|z_1||z_2| + i(\arg z_1 + \arg z_2 + 2\pi k) = \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 + 2\pi ik = \log z_1 + \log z_2 + 2\pi ik$.

Special values of the logarithm Consider $\log(1) = \ln(1) + i\arg(1) = 2\pi ik$. Also $\log(i) = \ln|i| + i\arg i = i\frac{\pi}{2} + 2\pi ik$ and $\log(-1) = i\pi + 2\pi ik$. Can also be shown that $\log(1/z) = -\log(z) + 2\pi ik$.

Principal branch of the logarithm Define $Log(z) \equiv \ln|z| + iArg(z) = \ln r + i\theta, -\pi < \theta \leq \pi$. We introduce a branch cut along the negative real axis. Hence we have that $Logz_1 = Logz_2 \implies z_1 = z_2$ because it is single-valued. Note, however, that $Log(z_1z_2) \neq Log(z_1) + Log(z_2)$ because $Arg(z_1z_2) \neq Arg(z_1) + Arg(z_2)$. $Logz$ is continuous and single valued only in the cut z-plane. Define $Log(-1 + 0i) = i\pi$ where $0i$ means just above the negative real axis and define $Log(-1 - 0i) = -i\pi$ where $-0i$ means just below the negative real axis.

nth branch Define the nth branch of the log: $Log_n z = \ln|z| + iArgz + 2\pi ni$, with $-\infty < n < \infty$. $n = 0$ is the principal branch. It is continuous and single-valued in $C\backslash(-\infty, 0]$ (this means a cut along the negative real axis from minus infinity to zero). Consider $Log_0(-r + i0) = \ln r + i\pi$. But note that $Log_1(-r - i0) = \ln r - i\pi + 2\pi i = \ln r + i\pi$. Hence Log_0 joins Log_1 along the negative real axis. This is true for the nth and the $(n+1)th$ branch. We can make all of the branches "fit" by defining $\log z$ on a domain in the cut z-plane for $Log_n z$ by taking it to be the nth copy of the z-plane. This forms the Riemann surface or $\log z$:



$\log z$ is continuous and single-valued on the Riemann surface.

Branch points The point $z = 0$ is a special point for $\log z$. For any curve that does not enclose the origin, the change in the argument (Arg) is zero. However, if the curve goes around the origin, then there is a finite change in the Argument. Call $z = 0$ a branch point. Then $\nexists$ closed curve around $z = 0$ on the Riemann surface of $\log z$.

Definition: Branch Point A branch point is a point in the z-plane, say z_0 such that $[f(z)]_c \neq 0, z_0 \in C, \forall c$ surrounding z_0.

Examples: Branch Point $z = 0$ is a branch point for $\log z$ and $z = 1$ is a branch point of $\log(z-1)$ and so on.

Branch point at infinity $z = \infty$ may be a branch point of $f(z)$, for example when you define $z = 1/\zeta$, then consider $g(\zeta) = f(1/\zeta)$ near $\zeta = 0$. If $\zeta = 0$ is a branch point of $g(\zeta)$ then $z = \infty$ is the branch point of $f(z)$.

Branches, Cuts, Branch Points and Single-Valued Branches $w = f(z)$ is single-valued for $z \in D$ if $\forall c \in D, [w]_c = 0, [f(z)]_c = 0$, where the square brackets refer to the change in the value of w on a curve around c. If not single-valued, then there exists a c for which $[w]_c \neq 0, [f(z)]_c \neq 0$. Generally, if $f(z) = |f|e^{i \arg f}$ then $f(z)$ is single-valued if $|f|$ is single-valued and $[\arg f]_c = 0, \forall c \in D$. The definition or specification of a branch of multivalued function $f(z)$ requires careful definition of D.

2.3 Wednesday 8 Oct 2014 Recitation

TA details Ben Wu, bhwu@caltech.edu

Property of ellipses Sum of distances to foci equals major axis.

2.4 Friday 10 Oct 2014

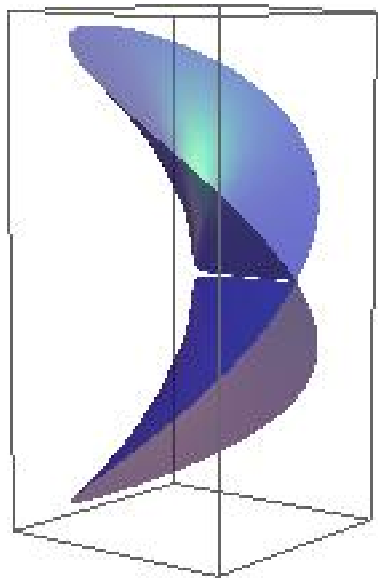
Behaviour at infinity Let $z = \frac{1}{\zeta}, g(\zeta) = f(1/\zeta)$, and examine $g(\zeta)$ near $\zeta = 0$. If $\zeta = 0$ is a branch point of $g(\zeta) \implies z = \infty$ is a branch point of $f(z)$.

Example of branch point at infinity Consider $w = \log z = \log(1/\zeta) = -\log \zeta$. We note that the branch point of $\log \zeta$ is at $\zeta = 0$. Hence $z = \infty$ is a branch point of $\log(z)$. Hence $\log z$ has two branch points: $z = 0$ and $z = \infty$.

Example 2 Define $w = (z^2 - 1)^{1/2}$. We note that there are branch points at $z = \pm 1$. Now define $z = 1/\zeta$. Then $w = (\frac{1}{\zeta^2} - 1)^{1/2} = \frac{(1-\zeta^2)^{1/2}}{\zeta} \approx \frac{1}{\zeta}$ near $\zeta = 0$. But there is no branch point at $\zeta = 0$, but just a pole. Hence w has no branch point at infinity.

Branches of log We note that we can define a valid branch of log as long as we do not include the branch points at $z = 0$ and $z = \infty$. Note that if we want to go near the branch point we need to introduce cuts to prevent a curve from going around the branch point. (i) Note further that cuts can be of any shape and don't need to be straight lines. (ii) Note also that cuts usually join two branch points of a function. In the log case, the cut connects the origin and infinity. However, there are exceptions. (iii) Branch points are common to all possible cuts. (iv) Branch points generally arise from $\log z, \log f(z), z^\alpha, \alpha$ not an integer.

Example: Fractional Power Let $w = z^{1/2}$. Check if $z = 0$ is a branch point. Take $w(re^{i0}) = r^{1/2}[\cos k\pi + i \sin k\pi]$. Check also $w(re^{2\pi i}) = r^{1/2}[\cos(k\pi + \pi) + i \sin(k\pi + \pi)] \neq w(re^{i0})$. Hence the function does not come back to the same point when θ is varied from 0 to 2π on a circle of radius r, hence $z = 0$ is a branch point. Now put $\zeta = 1/z = \rho e^{i\psi}$. Note that $w(\rho e^{i0}) = \frac{1}{\rho^{1/2}} \neq w(\rho e^{2\pi i}) = -\frac{1}{\rho^{1/2}}$. Hence $\zeta = 0$ is a branch point and hence $z = \infty$ is a branch point. Note that for any curve that goes around the branch point $z = 0$ will have $[w^{(1)}] = -2r^{1/2}$ and $[w^{(2)}] = 2r^{1/2}$ where $w^{(i)}$ refers to the ith branch with $k = i$. Note that since w is the square root, there are two distinct branches. In fact, for $w = z^{1/n}$, there will be n distinct branches. Note further that the two branches are connected to each other: $w^{(1)}(1) = 1, w^{(1)}(e^{2\pi i}) = -1, w^{(2)}(1) = -1, w^{(2)}(e^{2\pi i}) = 1$.



Multivalued functions may not have branch points For example $w = \log e^z = z + 2\pi ik$ does not have branch points. The Riemann surface is just a stack of discs.

Complex powers We first define the power $z^\alpha = e^{\alpha \log z}$ to be a multivalued function. The principal branch of this is going to be $z^\alpha = e^{\alpha Log z}$. By the definition of the log, we have that $z^\alpha = e^{\log(z^\alpha)}$. Hence we have that $\alpha \log z = \log z^\alpha$. For $(z^\alpha)^\beta = e^{\beta \log z^\alpha} = e^{\beta\alpha \log z} = z^{\alpha\beta}$. Now consider $z^\alpha z^\beta = e^{\alpha \log z} e^{\beta \log z} = e^{\alpha \log z + \beta \log z} = e^{(\alpha+\beta)\log z} = z^{\alpha+\beta}$. Furthermore, $(z_1 z_2)^\alpha = e^{\alpha \log(z_1 z_2)} = e^{\alpha(\log z_1 + \log z_2 + 2\pi ki)} = z_1^\alpha z_2^\alpha e^{2\pi kia}$.

Multivalued functions involving branch points Take $w = (z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$. which has branch points at $z = \pm 1$ and no branch point at infinity. Let $z - 1 = r_1 e^{i\theta_1}$ so $(z-1)^{1/2} = \sqrt{r_1} e^{i(\theta_1 + 2k_1\pi)/2}$ and $z + 1 = r_2 e^{i\theta_2}$ so that $(z+1)^{1/2} = \sqrt{r_2} e^{i(\theta_2 + 2k_2\pi)/2}$. Hence $w = \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2 + ik\pi}$, where $k = k_1 + k_2$. Hence this function has two branches, for $k = 0$ and $k = 1$. We can now define the branches. Define $0 \leq \theta_1 < 2\pi$ and $-\pi < \theta_2 < \pi$ for the branch with $k = 0$. Along the real axis to the left of $z = 1$, $\theta_1 = \pi, \theta_2 = 0$, and $w = i\sqrt{1-x^2}$. Above the real axis to the right of $z = 1$, $w = \sqrt{x^2-1}$ and slightly below the real axis, $w = -\sqrt{x^2-1}$. Hence w is discontinuous along the real axis to the right of $z = 1$. Hence we introduce a cut from $z = 1$ to infinity. Similarly, we introduce a cut from $z = -1$ to infinity. Note, however, that w is continuous along the $y - axis$. Hence we have a Riemann surface that is double-sheeted.

Chapter 3

Week 3

3.1 Monday 13 Oct 2014

Example Recall that if $w = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2} e^{i(\theta_1+\theta_2)/2+ik\pi}, k = 0, 1$. We can consider the angle a point in the domain makes with the two branch points ± 1 in two ways: (I): $0 \le \theta_1 < 2\pi, -\pi < \theta_2 \le \pi$ and (II): $-\pi < \theta_1 \le \pi, -\pi < \theta_2 \le \pi$. In (I), the function is continuous between -1 and 1 on the real axis, but discontinuous across the rest of the real axis. However, in (II), the function is discontinuous between -1 and 1 in the real axis but continuous across the rest of the real axis. Think of (I) as having the branch cut starting from -1, going to minus infinity, then to plus infinity, then decreasing to 1. Then branch cut for (II) is just between -1 to 1 across the origin. Hence we see that how the angles are defined will be related to how the branch cuts are conducted.

Making a single-valued branch for log[f(z)] Suppose we have $f(z)$ on some domain D in xy-space. Suppose that $f(z)$ is single-valued in D and we want to construct the single-valued branch of $w = \log f(z)$ which we call D' in uv-space. Suppose that D is simply connected. Hence $[f(z)]_c = 0, \forall c \in D$, the change in $f(z)$ on a curve around point c is zero. Further suppose that $f(z) \neq 0, \forall z \in D$, such that O' in uv-space is not in D'. There is hence no branch point associated with w. Also assume that $f(z) \neq \infty$ in D. If all these conditions hold, then generally $\log[f(z)]$ will be single-valued in D.

Exceptions to making single-valued branches for log[f(z)] Recall the general rules: (1). Branch points are generally associated with $\log[f(z)], \log z$ and z^α, with $\alpha \notin \mathbb{Z}$. (2) One can construct a single-valued branch of $\log[f(z)]$ provided the following conditions hold: (i) $f(z)$ has no zeroes on its domain and no singularities on its domain, ($f(z) \neq 0, \infty$) and (ii) either D is simply connected or D is not simply connected but $\log[f(z)]$ does not have a branch point at $z \to \infty$.

Hint for next week's homework Consider the function $w = \sin^{-1} z$. So $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$. Making the substitution $\zeta = e^{iw}$, the equation is a quadratic in ζ which can be solved for ζ and we can get $w = -i \log[iz + (1 - z^2)^{1/2}]$. Then we have the function $f(z) = iz + (1 - z^2)^{1/2}$ which has branch points where $(1 - z^2)^{1/2}$ has branch points. Pick the branch associated with the branch cut going from -1 through infinity to 1. This is a simply connected domain.

Another example Write $w(z) = q_\infty(\zeta e^{-i\alpha} + \frac{a^2}{4\zeta e^{-i\alpha}}) + \frac{\gamma}{4\pi i} \log \zeta$ where $\zeta = \frac{1}{2}(z^2 + (z^2 - a^2)^{1/2})^{1/2}$. This represents the complex potential $w(z) = \phi + i\psi$ for flow with velocity q_∞ with some angle of incidence α on a flat foil going from position $-a$ to a. We make the choice of circulation $\gamma = -2q_\infty \pi a \sin \alpha$ which streamlines the flow which has smooth separation of the trailing edge. This doesn't work, for some complex reason.

Limit Suppose we have some function $f(z)$ and we want the limit as $z \to z_0$. Then $\lim_{z \to z_0} = w_0$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $|z - z_0| < \delta$. For the limit to exist, the neighbourhood of w_0 must contain all of the nearby values of $f(z)$ in some full neighbourhood of z_0 (implies for all $\arg(z - z_0)$). You need to get the same value of the limit regardless of which direction you approach z_0 from. In general, delta is going to be a function of ϵ and z_0.

Limit at infinity Let $w = f(z)$ with $w_0 = \alpha$. Then $\lim_{z \to \infty} f(z) = \alpha$ means that $\forall \epsilon > 0, \exists \delta > 0$ such that $|z| > 1/\delta \implies |f(z) - \alpha| < \epsilon$ for all $\arg(z)$. This means that no matter how you go off to infinity, you must get the same limit value.

Example without limit at infinity Take $f = e^z$. As $z \to -\infty, e^z \to 0$ but as $z \to \infty, e^z \to \infty$. Hence e^z does not approach a constant and hence does not have a limit at infinity.

Properties of limits (i) $\lim_{z \to z_0}[f(z) + g(z)] = \alpha + \beta$ if $\lim_{z \to z_0} f(z) = \alpha$ and $\lim_{z \to z_0} g(z) = \beta$, (ii) $\lim_{z \to z_0} f(z)g(z) = \alpha\beta$ (ii) $\lim_{z \to z_0} f(z)/g(z) = \alpha/\beta$.

Exceptions to limits Functions may not be defined at $z = z_0$. For example, consider the function $f(z) = \frac{\sin z}{z}, z \neq 0$. In this case, $f(0)$ is defined by its limit so $f(0) = 1$.

Continuity $f(z)$ is continuous in the region R if for all $z_0 \in R$ if $\lim_{z \to z_0}$ is defined and equal to $f(z_0)$. If f is continuous, then $|f|$, $\bar{f}$, $Re(f), Im(f)$ are all continuous. However, the converse is not true. Counterexample: $f(z) = 1$ if z is rational, $f(z) = -1$ for z irrational. Then $|f| = 1$ is continuous but f is not continuous.

3.2 Wednesday 15 Oct 2014

Point at infinity Put $\zeta = \frac{1}{z}$ then $z = \infty$ corresponds to $\zeta = 0$. The ϵ neighbourhood of $z = \infty$ corresponds to the ϵ neighbourhood of $\zeta = 0$, and continuity of $f(z)$ at $z = \infty$ corresponds to continuity of $f(1/\zeta)$ at $\zeta = 0$.

Examples Consider e^z. Consider $z = \frac{1}{\zeta}$. Note that $e^{1/\zeta}$ is not continuous at $\zeta = 0$.

Differentiability A function f defined on D is said to be differentiable at $z = z_0$ in D if $\lim_{z \to z_0, \forall \arg(z - z_0)} \frac{f(z) - f(z_0)}{z - z_0}$ exists. Define this to be $f'(z_0)$. If $w = f(z)$, then call this $\frac{dW}{dz}\Big|_{z=z_0}$. We note that we can write any $f(z) = \frac{f(z)-f(z_0)}{z-z_0}(z - z_0) - f(z_0)$. Then if the limit of the fraction exists as $z \to z_0$, then the function $f(z) \to f(z_0)$ and hence f is continuous. Hence $f(z)$ is differentiable implies $f(z)$ is also continuous. Note that we can write $z - z_0 \equiv h = |h|e^{i\phi}$, so we write $f'(z) = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$. Since the limit is independent of the direction, this limit must apply for all ϕ. If the limit exists for all $z \in R$, then we say that $f'(z)$ exists in R, so $f(z)$ is differentiable everywhere in R.

Examples Consider $w = z^n, n > -0$. Then $\frac{(z+h)^n - z^n}{h} = \frac{z^n + hnz^{n-1} + \ldots - z^n}{h}$ which equals nz^{n-1} when $h \to 0$.

Chain rule Consider $F(g(z))$. Then $\frac{dF}{dz} = \frac{dF}{dg}\frac{dg}{dz}$.

Example 1 Calculate the derivative of $f(z) = \bar{z}$. Then $\frac{f(z+h)-f(z)}{h} = \frac{\bar{z}+\bar{h}-\bar{z}}{h} = \frac{\bar{h}}{h}$. But since $h = |h|e^{i\phi}$ so this is equal to $e^{-2i\phi}$. But this is not independent of the direction of approach ϕ. For instance, if $\phi = 0$, then the limit becomes $e^0 = 1$. But if $\phi = \pi/2$, then the limit is $e^{-i\pi} = -1$. But we have that the limit must exist for all directions of approach for the function to be differentiable. Hence $f(z)$ is not differentiable, even though its real and imaginary parts are differentiable everywhere.

Example 2 Consider e^z. Then $\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} = \frac{e^h - 1}{h} e^z$. Put $h = a + ib$, and note that $e^a = 1 + a + O(a^2)$ while $e^{ib} = \cos b + i \sin b = (1 - \frac{b^2}{2} + \ldots)(b + O(b^3))$. Then we have that $e^h - 1 = e^a e^{ib} - 1 = (1 + a + O(a^2))(1 + ib + O(b^2)) - 1 = a + ib + O(a^2, b^2) = h + O(h^2)$ so $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, which is independent of direction.

Test for differentiability: Cauchy-Riemann Equations Write $f(z) = u(x, y) + iv(x, y)$. Take h to be real. Then $f(z + h) = u(x + h, y) + iv(x + h, y)$. Then take $\frac{f(z+h)-f(z)}{h} = \frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}$. We want the limit as $h \to 0$. But this is the ordinary limit of functions u and v. Hence we have that the limit as $h \to 0$ is just $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. Now we consider $h = ik$ and approach along the imaginary direction. Then we have that $\frac{f(z+ik)-f(z)}{ik} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$. Now we have that the limit must be independent of direction. Hence these two expressions must be equal. Hence we obtain that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. We should actually do it for any direction, but this gives the right answer anyway. Write these Cauchy-Riemann equations as $u_x = v_y, u_y = -v_x$.

Facts about the C-R Equations (1) The C-R equations are necessary for differentiable functions. (2) The C-R equations are NOT SUFFICIENT for differentiable functions. To make the C-R equations sufficient, we require that (2a) u_x, u_y, v_x, v_y are continuous at the point you are interested in and (2b) the C-R equations must be satisfied in an ϵ neighbourhood of that point.

Summary of differentiability $w = f(z)$ is differentiable (i.e. $\frac{dW}{dz}$ exists) iff $u_x = v_y, u_y = -v_x$. Then write $\frac{dW}{dz} = f'(z) = u_x + iv_x = u_x - iu_y$ and so on.

Analyticity A function $f(z)$ is said to be analytic at a point z_0 if it is differentiable in some ϵ neighbourhood of z_0. Note that (1) $f(z)$ is analytic in the region R if it is analytic at every point. (2) On domains (which we recall cannot contain its boundary points), differentiability and analyticity are the same. But not so for regions.

Singularity If $f(z)$ is not analytic at $z = z_0$, then we say that z_0 is a singularity of $f(z)$. For example, at branch points, the function is not analytic.

3.3 Recitation Wednesday 15 Oct 2014

Riemann Sphere Stereographic projection of the complex plane onto a sphere. South pole is zero, north pole is infinity. Equator is complex numbers with magnitude 1. Lower hemisphere is complex numbers of magnitude less than 1. Upper hemisphere has complex numbers greater than magnitude 1. Nothing inside the sphere! Only on the surface.

3.4 Friday 17 Oct 2014

Orthogonality of level sets Let$f(z) = u(x,y) + iv(x,y)$. Also let $u(x,y) = c$, c constant. Then we can write the level sets of $u(x,y)$ and we can form the gradient $\nabla u(x,y) = u_x\hat{i} + u_y\hat{j}$. We know that the gradient is always orthogonal to the level sets. Similarly, for $v(x,y) = c'$, c' constant, we can construct the gradient $\nabla v(x,y) = v_x\hat{i}+v_y\hat{j}$ which will be orthogonal to the level sets of $v(x,y)$. Take some point at the intersection of one level set of u and another of v. Then the dot product of their gradients is $u_xv_x + u_yv_y$. But if $f(z)$ satisfies the CR equations, we know that this is going to be equal to $v_yv_x - v_xv_y = 0$. Hence we have that the gradients must be orthogonal to each other, and hence the level sets of u and v are always orthogonal to each other. Hence the families of of contours/level sets $u(x,y) =$ constant and $v(x,y) =$ constant are orthogonal sets.

Constant components implies constant function If $f(z)$ has constant u, constant v or constant $u^2 + v^2$, then $f(z)$ is a constant. If u is a constant, we have that $Re(f) =$ constant, so $u_x = 0 = v_y$ by the CR equations so $v = g(x)$ for some function g. But we also have that $u_y = 0 = -v_x$ so $g(x)$ =constant. Hence v is also a constant. Taken together, $f(z)$ is a constant. Repeat same argument if v is constant. Now if $|f|^2$ is a constant, we have that $u^2 + v^2$ =constant. Hence, differentiating with respect to x, we have that $uu_x+vv_x = 0$, and differentiating with respect to y, we have that $uu_y+vv_y = 0$. But with the CR equations, we have that $uu_x - vu_y = 0$ and $uu_y + vu_x = 0$. In matrix form, $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0$. So if $u^2 - v^2 \neq 0$ then the matrix is invertible and this requires that $u_x = u_y = 0$. Hence invoke the same argument to claim that $f(z)$ is constant.

Entire function An entire function is analytic in the whole of the finite z-plane. We have that $|z| < \infty$. Then this function has no singularities anywhere in the finite z-plane. Example: all polynomials, $e^z, \sin z, \sinh z$ are entire. Note that $\log(z)$ is not entire, but it is analytic everywhere in the finite z-plane except at the branch points.

Complex Logarithm Recall that $\log(z) = \ln|z| + i\arg(z) = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x) + 2\pi ik$. Write $u(x,y) = \ln\sqrt{x^2+y^2}$ and $v(x,y) = i\tan^{-1}(y/x)$. Then $u_x = \frac{x}{x^2+y^2}$ and $v_y = \frac{1}{x}\frac{1}{1+y^2/x^2} = u_x$. Also, $u_y = \frac{-y}{x^2}\frac{1}{1+y^2/x^2}$ and $v_x = \frac{y}{x^2+y^2}$ so $u_y = -v_x$ and the CR equations are satisfied. We hence can write $\frac{d(\log z)}{dz} = u_x + iv_x = \frac{x-it}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$. Note that $\frac{1}{z}$ does not have a branch point at $z = 0$, but has a pole instead. We note that if $f(z) = \log z$ then f is not single valued in the z-plane. But $f'(z) = \frac{1}{z}$ is single-valued in the z-plane with a singularity at $z = 0$.

Powers Consider $f(z) = e^\alpha$ where alpha is not an integer. Then write $\frac{dz^\alpha}{dz} = \frac{d}{dz}e^{\alpha\log z} = e^{\alpha\log z}\frac{d}{dz}(\alpha\log z) = \frac{\alpha}{z}e^{\alpha\log z} = \alpha z^{\alpha-1}$. Note that if α is not an integer then $f(z)$ has a branch point at $z = 0$, and hence its derivative also has a branch point there. We need to ensure that we evaluate the derivative on the same branch! Use the same branch as the original function.

Harmonic Functions Take $f(z) = u(x,y) + iv(x,y)$. By CR, $u_x = v_y, u_y = -v_x$. Differentiate the first w.r.t y and the second w.r.t x, obtaining $u_{xy} = v_{yy}, u_{yx} = -v_{xx}$. If $u \in C^2$, then the mixed second partial derivatives are equal and hence $v_{yy} = -v_{xx}$ or $\nabla^2 v = 0$. Repeating this by changing the variable of differentiation, we have that $\nabla^2 u = 0$. Both u and v are hence harmonic functions. Note that the converse is also true: If $\nabla^2 u = 0$ then there exists an analytic $f(z)$ such that $u = Re(f)$. Same for $\nabla^2 v$.

Harmonic Conjugate A harmonic function $v(x,y)$ related to $u(x,y)$ by the CR equations is said to be the harmonic conjugate of $u(x,y)$. Notes: If the harmonic conjugate exists, it is unique up to an additive constant. Note that if $v(x,y)$ is the harmonic conjugate of $u(x,y)$, then $-u(x,y)$ is the harmonic conjugate of $v(x,y)$. Note the minus sign! This is because $if(z) = iu - v = -(v - iu)$. We say that u and v are conjugate pairs if they satisfy the CR equations.

Finding the harmonic conjugate Given $u(x,y)$, find $v(x,y)$ such that $v(x,y)$ is the harmonic conjugate of $u(x,y)$ in some domain D. We construct $v(x,y)$. Suppose that D has the following properties: Take the point $(x,y) \in D$ such that $u(x,y)$ satisfies Laplace's equation at that point. Then take a point $(x_0,y_0) \in D$ such that we can connect these points using an L-shaped curve. We first integrate $v_x = -u_y$ with respect to x to obtain: $v(x,y) = -\int_{x_0}^{x} u_y(x',y)dx'+c(y)$. We then take the second equation $v_y = u_x$ and form the derivative: $v_y = -\frac{\partial}{\partial y}\int_{x_0}^{x} u_y(x',y)dx'+c'(y) = \int_{x_0}^{x} -u_{yy}(x',y)dx'+c'(y)$ after taking the derivative inside the integral. Because u is harmonic, we can write the RHS as $\int_{x_0}^{x} u_{x'x'}dx' + c'(y) = u_x(x,y) - u_x(x_0,y) + c'(y)$. But we know that by the CR equations, this is going to equal $u_x(x,y)$. Hence we have that $c'(y) = u_x(x_0,y)$. Hence $c(y) = \int_{y_0}^{y} u_x(x_0,y')dy' + v(x_0,y_0)$. Hence we have $v(x,y) = -\int_{x_0}^{x} u_y(x',y)dx' + \int_{y_0}^{y} u_x(x_0,y')dy' + v(x_0,y_0)$.

Related results Suppose u is harmonic in D. then $g(z) = u_x(x,y) - iu_y(x,y)$ satisfies the CR equations because $(u_x)_x = (-u_y)_y$ because u is harmonic. Also, $(u_x)_y = -(-u_y)_x$ because u is C^2. We hence have that $g(z)$ is analytic in D and if u is a real part of analytic $f(z)$ where $f(z) = u + iv$, then $g(z) = f'(z)$.

Finding branch points Construct the positions of all complex points relative to suspected branch points, then examine what happens when you go around that point.

Chapter 4

Week 4

4.1 Monday 20 Oct 2014

Example: Finding the complex conjugate Let $u(x,y) = x^3 + axy^2$. So $\nabla^2 u = 6x + 2ax = 0$ if $a = -3$. Hence $u = x^3 - 3xy^2$ and $u_x = 3x^2 - 3y^2 = v_y$ by Cauchy-Riemann equations. Hence we integrate this with respect to y to obtain $v = 3x^2y - y^3 + D(x)$ for some unknown function $D(x)$. But we have the other Cauchy-Riemann equation that $u_y = -6xy = -v_x$. Hence we differentiate v by x to obtain $6xy + D'(x) = 6xy$. So $D'(x) = 0$ and $D(x)$ is a constant. Hence we have the harmonic conjugate $v = 3x^2 - y^3 + \epsilon$, where ϵ is a constant. Then $f = u + iv = (x+iy)^3 + \epsilon = z^3 + \epsilon$ is an analytic function.

Alternative method to find the complex conjugate Construct $u_x - iu_y = g(z)$ which is the derivative of some analytic function $f = u + iv$ for unknown v. So $g(z) = (3x^2 - 3y^2) + 6ixy = 3z^2 = f'(z)$. So $f = z^3 + \epsilon$ immediately.

Conformal Map Consider $w = f(z)$ as a mapping. If $f(z)$ is analytic, then w is a conformal mapping from some domain D of the xy-plane to some domain $\mathcal{D}$ in the uv-plane. A mapping is said to be conformal at every point at which its derivative $f'(z)$ is non-zero and non-infinite. Suppose we have two curves in the xy plane C_1 and C_2. We consider the mapping of these curves onto the w-plane C_1' and C_2'. The mapping is said to be conformal if the angle between the two curves at a point in the xy-plane is equal to the angle between the two images in the uv-plane.

Inverse function theorem We write the inversion of $w = f(z)$ to be $z = f^{-1}(w)$. Then the IFT states that provided $f(z)$ is analytic, then $z = f^{-1}$ exists and is analytic provided $f'(z) \neq 0$. Sketch of proof: take the local Taylor expansion $w - w_0 = (z - z_0)f'(z_0) + O(z - z_0)^2$ so $z - z_0 = \frac{w - w_0}{f'(z_0)}$+other terms. Hence locally, $\frac{dz}{dw}\Big|_{z=z_0, w=w_0} = \frac{1}{f'(z)}$. Note that in the inversion we now have $x(u,v)$ and $y(u,v)$, inverting the dependence of the function on its inputs. We note that the CR equations hold in the sense that $\left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{\partial y}{\partial v}\right)_u$ and $\left(\frac{\partial x}{\partial v}\right)_u = -\left(\frac{\partial y}{\partial u}\right)_v$.

Relating partial derivatives under an inversion Consider $u(x,y), v(x,y)$. Now we want to know what the partial derivatives of $x(u,v)$ and $y(u,v)$ are. We write $\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial v}{\partial x}\right)_y & \left(\frac{\partial v}{\partial y}\right)_x \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$. Then we can write the column vector $(dx, dy)^T$ in terms of some matrix multiplied by $(du, dv)^T$, which will just be the inverse of the 2x2 matrix. But we also can write the total differential of $(dx, dy)^T$ in terms of the matrix of the partials of $x(u,v)$ and $y(u,v)$ multiplied by $(du, dv)^T$. We hence equate these two expressions for $(dx, dy)^T$ to obtain the Cauchy-Riemann equations. It can also be shown that the level sets of $x(u,v)$ and $y(u,v)$ will be orthogonal to each other at every point.

Complex Integration Recall the definition of integration for real variables: $\int f(x)dx$ exists as a Riemann integral. Recall also the fundamental theorem of integration for real variables: Take $f(x)$ to be continuous on integral $[a,b]$ and has antiderivative $F(x)$ such that $F'(x) = \frac{dF(x)}{dx} = f(x)$. Then $\int_a^b f(t)dt = F(a) - F(b)$. Let $\phi(t)$ be a parametrised complex function of a real variable: $\phi(t) = \phi_1(t) + i\phi_2(t)$ where ϕ_1 and ϕ_2 are real-valued continuous function of t. Take ϕ_1 and ϕ_2 to be differentiable with derivatives $\dot{\phi}(t) = \dot{\phi}_1(t) + i\dot{\phi}_2(t)$. Also let ϕ_1, ϕ_2 be integrable on $a \leq t \leq b$. Then the integral of the complex function $\int_a^b \phi(t)dt = \int_a^b \phi_1(t)dt + i\int_a^b \phi_2(t)dt$. Also from the real valued fundamental theorem of calculus we have $\frac{d}{dt}\int_a^t \phi(u)du = \phi(t)$ and $\int_a^b \dot{\phi}(t)dt = [\phi(t)]_a^b$.

Curves in the complex plane Define $x = \xi(t), y = \eta(t)$ such that $z = x + iy$ is parametrised by $\zeta(t) = \xi(t) + i\eta(t)$. Then $z = \zeta(t)$ describes a curve C in the complex plane. Call the trace of C to be the set of points occupied by C. A simple curve does not cross itself. A closed curve has endpoints that are equal. Call an **"arc"** a continuously differentiable curve on some interval. Take $ArcC_j$ to be the curve $z = \zeta_j(t)$ for some $a_j \le t \le b_j$. A set of arcs form a **contour** if the ending of one arc is joined to the beginning of the next arc: $\zeta_j(b_j) = \zeta_{j+1}(a_j)$. Hence a contour can be written as a set of arcs. Note that the derivatives of the arcs do not need to match at the joining points.

Complex function on C Let $f(z), z = x + iy$ be a complex function, and curve C with $z = \zeta(t)$. Then we have $\phi(t) = f[\zeta(t)], t \in [a, b], t \in \mathbb{R}$ for the complex value of the function on C. Then $\dot{\phi}(t) = \left.\frac{df(z)}{dz}\right|_{z=\zeta(t)} \frac{d\zeta(t)}{dt}$ by the chain rule.

Integral on a curve Consider a curve defined by $x = \zeta(t), y = \eta(t)$. $\int_C f(z)dz = \int_C [u(x, y) + iv(x, y)](dx + idy) = \int_a^b [u + iv](\dot{\zeta} + i\dot{\eta})dt = \int_a^b f(\zeta(t))\frac{d\zeta(t)}{dt}dt$.

4.2 Wednesday 22 Oct 2014

Complex integration Recall that $\int_c f(z)dz = \int_a^b f(\zeta(t))\frac{d\zeta}{dt}dt$ where $\zeta(t) = \xi(t) + i\eta(t)$.

Arclength Define $ds = \sqrt{dx^2 + dy^2} = dt\sqrt{\dot{\xi}^2 + \dot{\eta}^2} = \left|\frac{d\zeta}{dt}\right| dt$. So integrating along arclength is just $\int_c f(z)ds = \int_a^b f(\zeta(t)) \left|\frac{d\zeta}{dt}\right| dt$.

Conjugate integration Take $\int_c f(z)d\bar{z} = \int_a^b f(\zeta(t))\frac{d\bar{\zeta}(t)}{dt}dt = \int_a^b (u + iv)(dx - idy)$.

Properties of integrals

- Integration is linear: $\int_c (\alpha f(z) + \beta g(z))dz = \alpha \int_c f(z)dz + \beta \int_c g(z)dz$
- Consider a contour $C = \{C_1, C_2, \ldots, C_n\}$. Then $\int_C f(z)dz = \sum_{j=1}^n \int_{C_j} f(z)dz$. If C is closed, write $\oint_C f(z)dz$.

Theorem Let $F = U + iV$ be analytic and single-valued in some domain D. Also let $f = \frac{dF}{dz} = F'$ with f continuous. Also let $c \in D, x = \xi(t), y = \eta(t), z = \zeta(t) = \xi(t) + i\eta(t)$. Write $f = F' = U_x + iV_x = V_y - iV_y$ by CR equations. Also, $dz = dx + idy = (\dot{\xi} + i\dot{\eta})dt$. Consider $\dot{U}(x, y)$ using the chain rule to obtain $\dot{U}(x, y) = U_x\dot{\xi}(t) + U_y\dot{\eta}(t)$ and $\dot{V}(x, y) = V_x\dot{\xi}(t) + V_y\dot{\eta}(t)$. We can write $fdz = (U_x + iV_x)(dx + idy) = (U_x + iV_x)(\dot{\xi} + i\dot{\eta})dt = (U_x\dot{\xi} - V_x\dot{\eta}) + i(V_x\dot{\xi} + U_x\dot{\eta})$ after gathering real and imaginary parts. Using the CR equations, we can simplify this to become: $(\dot{U} + i\dot{V})dt$. This is just $(\frac{dU}{dt} + i\frac{dV}{dt})dt$. Hence we have a real integral $\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} F'(z)dz = \int_a^b \left(\frac{dU}{dt} + i\frac{dV}{dt}\right) dt, z_1 = \zeta(a), z_2 = \zeta(b)$. But using the fundamental theorem of calculus, we have that this is just $[U + iV]|_a^b = F(z_2) - F(z_1)$. Hence we have that $\int_{z_1}^{z_2} F'(z)dz = F(z_2) - F(z_1)$ and the integral only depends on the end-points. If C is closed, then the integral is zero (note that F has to be single-valued). **Note that the converse is also true!** If $\oint_c f(z)dz = 0$ and f is continuous in D, then $\exists F(z)$ such that $F' = f$.

Notes on the previous theorem The antiderivative F must be analytic. Also, if f is a branch, the F' must be on the same sheet of the Riemann surface.

Cauchy's Integral Theorem Suppose f is analytic and single-valued in D. Let D be simply connected. Then:

$$\oint f(z)dz = 0, \forall c \in D$$

Write $f = u + iv$. Consider the function $G(\lambda) = \lambda \oint f(\lambda z)dz, 0 \le \lambda \le 1$. Then $G(1) = \oint f(z)dz$ and $G(0) = 0$. Differentiating, we obtain $\frac{dG}{d\lambda} = \oint f(\lambda z)dz + \lambda\frac{d}{d\lambda}\oint f(\lambda z)dz = \oint \left(f(\lambda z) + \lambda\frac{df(\lambda z)}{d(\lambda z)}\frac{d(\lambda z)}{d\lambda}\right) = \oint \frac{d}{dz}(zf(\lambda z))dz$. But since f is single-valued in D, the integral is going to be zero. Hence $\frac{dG}{d\lambda} = 0$ so G is a constant. But since $G(0) = 0$, this requires that $G(1) = 0$ also. Hence $\oint f(z)dz = 0$.

Doubly-connected example where the Cauchy Integral Theorem fails Let $f(z) = 1/z$ and let D be the annulus $\frac{1}{2} < |z| < 2$. On $|z| = 1$, let $z = e^{i\theta}, dz = ie^{i\theta}d\theta$ so $f(z) = e^{-i\theta}$ and $fdz = e^{i\theta}ie^{-i\theta}d\theta = id\theta$. Hence $\oint_c fdz = i\int_0^{2\pi} d\theta = 2\pi i \ne 0$. Note that $f = 1/z$ has antiderivative $F(z) = \log z$ which is not single-valued in D. Hence the Cauchy integral theorem fails in this case.

Green's theorem Consider the imaginary part of the integral $Im[\oint_C fdz] = \oint_C (udy + vdx) = \iint_A \left(\frac{du}{dx} - \frac{dv}{dy}\right) dxdy$ by Green's Theorem. The real part is given by $Re[\oint_C fdz] = \oint_C (vdy - udx) = \iint_A \left(\frac{dv}{dx} + \frac{du}{dy}\right) dxdy$. Then we have that

$\oint f dz = Re[\oint f dz] + iIm[\oint f dz]$. But the Cauchy Riemann equations imply that the double integrals are going to be zero. But f has to be analytic everywhere on the domain for the double integrals to vanish.

Integration on circles Consider the curves $z = Re^{i\theta}$ or $z - z_0 = Re^{i\theta}$. Then $dz = iRe^{i\theta} d\theta = iR\frac{z}{R} d\theta = izd\theta$.

Bounds on integrals For F complex, $t \in \mathbb{R}, a \le t \le b$.

$$\left| \int_a^b F(t) dt \right| \le \int_a^b |F(t)| dt$$

Proof Suppose some real-valued function $g(t) \le G(t), a \le t \le b$. Then $\int_a^b g(t) dt \le \int_a^b G(t) dt$. Now we let $\int_a^b F(t) dt = Je^{i\theta}$ be some complex number. Now J is real. Hence we can write $J = \int_a^b e^{-i\theta} F(t) dt = \int_a^b Re[e^{-i\theta} F(t)] dt$ because J is real. Let $G(t) = |F(t)|, g(t) = Re[e^{-i\theta} F(t)]$ be two real-valued functions. We note that $g(t) \le G(t)$. Hence $J \le \int_a^b |F(t)| dt$, and the bound on the integral follows immediately.

Another bound on the integral

$$\left| \int_c f(z) dz \right| \le ML$$

where M is the maximum of the $|f(z)|$ on C and L is the length of C. Note dz here, not dt.

Proof of the second bound Let $f = \rho e^{i\phi}$, where $\rho(s), \phi(s)$ are functions of arc length s. Then we have $dz = dx + idy = ds \cos\theta + ids \sin\theta = ds \cdot e^{i\theta(s)}$. Then $\left|\int_C f(z) dz\right| = \left|\int \rho(s) e^{i(\theta(s) + \phi(s))} ds\right|$. But we have that this is going to be less than or equal to $\int \rho(s) ds = \int |f| ds$, which is less than or equal to $|f|_{max} \int ds = |f|_{\max} L$.

4.3 Recitation 22 Oct 2014

CR for Polar Coordinates Write $f(r, \theta) = u(r, \theta) + iv(r, \theta)$. Then the CR equations are:

$$ru_r = v_\theta$$
$$u_\theta = -rv_r$$

CR Theorem for Polar Coords If the first order partial derivatives of u and v exist in the neighbourhood of some point z_0, and they satisfy the Cauchy-Riemann equations, then $f'(z_0)$ exists and equals $f'(z_0) = e^{-i\theta}(u_r + iv_r)$.

Poles in polynomials/rational functions Consider $P_n(z) = (z - z_1)(z - z_2)\cdots$. Note that $z = \infty$ is a pole. For $\frac{1}{P_n(z)}$ the poles will be at $z_1, z_2, \cdots$ etc.

Making branch cuts Note that when you construct branch cuts, it is the combined angle (not the angle with respect to individual points) that should be discontinuous.

Chapter 5

Week 5

5.1 Monday 27 Oct 2014

Recall Cauchy-Integral Theorem Consider $f(z)$ analytic and single-valued in D, with D simply connected. Then we have $\oint_C f(z)dz = 0$ for any contour C in D.

Corollary to Cauchy-integral Theorem Suppose $f(z)$ is analytic and single-valued in D. Then there exists a function $F(z)$ also analytic in D such that $F' = f$. Note that $F(z)$ may not be single-valued.

Sketch of proof of Corollary Consider the function $F(x) = \int_{z_0}^{z} f(\zeta)d\zeta$ along some curve. Consider $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right|$ where h is complex. It will suffice to show that as $h \to 0$, then $F'(z)$ goes to $f(z)$. Write $\left|\frac{1}{h}\int_z^{z+h} f(\zeta)d\zeta - \frac{1}{h}\int_z^{z+h} f(z)d\zeta\right|$. Note the second term is just $f(z)$ because $f(z) = f(z)\frac{h}{h} = f(z)\frac{1}{h}\int_z^{z+h} d\zeta$. Hence we can simplify the expression to $\left|\frac{1}{h}\int_z^{z+h}[f(\zeta) - f(z)]d\zeta\right|$. But by the bounds on the integral, we know that the expression is going to be less or equal to $\max_{|\zeta - z|<h} |f(\zeta) - f(z)|$. Since $f(z)$ is analytic and hence differentiable, we can bring h arbitrarily close to zero and show that F exists such that $F' = f$ in D.

What happens if the domain is not simply connected? Suppose D is multiply connected. Then F exists but is not single-valued. For example, take $f(z) = 1/z$ and $F(z) = \log z$. Take the domain to be the complex plane without the origin. Suppose we want to evaluate $\int_1^z \frac{1}{z'}dz' = \int_1^z \frac{d}{dz'}\log z' dz' = \ln|z| + i\theta + 2\pi i N, N \in \mathbb{Z}$. This integral depends on the path! The more times you wind around the origin, the larger N is. Call N the winding number. Then we can write $\int_1^z \frac{dz'}{z'} = Logz + 2\pi i N$. To make this integral single-valued, we need to introduce a cut and make the domain simply-connected. Because of the cut, the curve cannot go around the origin. Then Cauchy's Integral Theorem holds.

Contour Deformation Pick the usual f and domain D such that Cauchy's integral theorem holds. Example 1. Define two points A and B. Pick two contours that connect $A \to B$, C_1 and C_2. We know that $\oint_{C_1 - C_2} f(z)dz = 0$ for analytic f by Cauchy's integral theorem. But we know that by reversing the direction of the curve we change the sign of the integral. Then we have $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$, the integral is independent of path provided there is no singularity of $f(z)$ on the domain between C_1 and C_2.

Contour Deformation Example 2 Consider two closed curves C_2 and C_1 such that C_2 is completely contained in the interior of C_1. Let $f(z)$ be analytic between C_2 and C_1. Connect the contours with a straight line L_1 from the outside to the inside. Define a composite contour $C = C_1 + L_1 + (-C_2) + (-L_1)$. Then $\oint_C f(z)dz = 0$ by the Cauchy Integral Theorem. Then we can write this as $\int_{C_1} + \int_{L_1} + \int_{-C_2} + \int_{-L_1} = 0$ (short-hand). Hence we have that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$. This means that we can deform closed contours in any way we want (while remaining within the domain where $f(z)$ is analytic) while not changing the value of the integral on the contour.

Contour Deformation with singularities Consider a function with N singularities, but being analytic everywhere else on a domain. Then we have that $\int_C f(z)dz = \sum_N \int_{C_i} f(z)dz$, where C_i is a small curve/circle around the ith singularity.

Cauchy's Integral Formula Let f be analytic in D and consider some curve $C \in D$. Then $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz = \begin{cases} f(z_0), & z_0 \in C \\ 0, & z_0 \notin C \end{cases}$. The value of a function at an interior point is completely determined by the value of the function on any curve surrounding it, provided that the function is analytic in the domain. In particular, when $f(z) = 1$,

$\frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}dz = \begin{cases} 1, & z_0 \in C \\ 0, & z_0 \notin C \end{cases}$.

Proof We note that $\frac{f(z)}{z-z_0}$ is analytic in C if $z_0 \notin C$ so that the denominator does not vanish. Hence the Cauchy Integral theorem gives us $\oint \frac{f(z)}{z-z_0}dz = 0$ immediately. If $z_0 \in C$, then we just deform the contour C into a small circle of radius δ surrounding z_0, the singularity. Then we have $\frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\oint_{|z-z_0|=\delta} \frac{f(z)}{z-z_0}dz = \frac{f(z_0)}{2\pi i}\oint_{|z-z_0|=\delta} \frac{1}{z-z_0}dz + \frac{1}{2\pi i}\oint_{|z-z_0|=\delta} \frac{f(z)-f(z_0)}{z-z_0}dz$. We write $dz = i\delta e^{i\theta}d\theta$, so that $\oint \frac{dz}{z-z_0} = \oint \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}d\theta = 2\pi i$. Also, we also know that $f(z)$ is continuous at $z = z_0$. Hence we have that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$ by continuity. By the integral bounds, we have that $\left|\oint \frac{f(z)-f(z_0)}{z-z_0}dz\right| \leq \frac{\epsilon}{\delta}2\pi\delta = 2\pi\epsilon$ which goes to zero as $\epsilon \to 0$. Hence we have that $\frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz = f(z_0)$ if $z_0 \in C$.

5.2 29 Oct 2014

Generalization of Cauchy's Integral formula to derivatives Recall that $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & z_0 \in C \\ 0, & z_0 \notin C \end{cases}$, $f(z)$ analytic in some domain D. Then its derivatives exist and are themselves analytic. Consider the function $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z}d\zeta$. Then we evaluate $\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi ih}\oint_C \frac{f(\zeta)}{\zeta-z-h}d\zeta - \frac{1}{2\pi ih}\oint_C \frac{f(\zeta)}{\zeta-z}d\zeta$. We can combine the integrals to obtain $\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)}d\zeta$. We now take the limit as $h \to 0$. Then we get $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^2}d\zeta$. We can generalise this by induction to obtain that $f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$. Hence if we know $f(\zeta)$ on a curve C, we can construct the function and all its derivatives. Hence an analytic function is infinitely differentiable.

Morera's Theorem Let D be a domain, not necessarily simply connected. Also let $f(z)$ be continuous at each $z \in D$. Suppose $\oint_C f(z)dz = 0$ for all $C \in D$. Then $f(z)$ is analytic.

Sketch of proof Recall that $\oint_C f dz = 0 \implies \exists F(z)$ such that $F' = f$. But this means that F is analytic. But if F is analytic, then its derivative $F' = f$ is also analytic (from the Cauchy Integral Formula).

Cauchy Inequality Suppose $f(\zeta)$ is analytic for $|\zeta - z| < R$, in the interior of some circle. Let M_R be the maximum value of $|f(\zeta)|$ on the circle $|\zeta - z| = R$. Then the maximum value of the derivative inside the circle $|f^{(n)}(z)| \leq \frac{n!M_R}{R^n}$ from the Cauchy Integral formula and bounds on the integral. In particular, for $n = 1$, we have that $|f'(z)| \leq \frac{M_R}{R}$.

Liouville's Theorem Suppose $|f(z)| < M$ in the whole z-plane (including infinity). Assume $f(z)$ is analytic. Then f has to be a constant.

Sketch of Proof Observe that from Cauchy's Inequality $|f'(z)| \leq \frac{M_R}{R}$. But now we can take $R \to \infty$ to include the whole z-plane. Hence we have that $|f'(z)| = 0$, and hence $f(z)$ is constant.

Entire Function If $f(z)$ is analytic for $|z| < \alpha$ on the finite z-plane (no singularities), then $f(z)$ is said to be entire. Note that from Liouville's theorem, the entire function will be unbounded at infinity if it is not the constant function. But functions with singularities in the finite z-plane can be bounded at infinity.

Point at Infinity Say that $f(z)$ is analytic at infinity if $g(\zeta) = f(1/\zeta)$ is analytic at $\zeta = 0$.

Fundamental Theorem of Algebra Consider a polynomial of degree N: $P_N(z) = a_0 + a_1 z + \ldots + a_N z^N$, $a_N \neq 0$, $a_i \in \mathbb{C}, \forall i$. We show that $P_N(z)$ has at least one zero for $N \geq 1$. That is, $\exists z_0$ such that $P_N(z_0) = 0$. We proceed by contradiction. Suppose we do not have any zeros. Then $f(z) = \frac{1}{P_N(z)}$ is clearly entire. We also know that it will be bounded because $|P_N(z)| \to |a_N||z^N|$ as $z \to \infty$ for all $\arg z$ and $|z^N|$ is unbounded as $z \to \infty$. (Note that this does not work for $f(z) = e^z$ because $|e^{-z}| \to 0, x \to \infty$ and $|e^{-z}| \to \infty, x \to -\infty$. e^{-z} has an essential singularity at infinity.). Hence $\frac{1}{|P_N(z)|} \to 0$ as $z \to \infty$. Hence $\frac{1}{P_N(z)}$ is bounded and entire. But the Liouville Theorem says that the only function that satisfies this is the constant function. But we know that $\frac{1}{P_N(z)}$ cannot be a constant. Contradiction. Hence $P_N(z)$ has at least one zero. Call this point z_0. Now we write $P_N(z) = (z - z_0)P_{N-1}(z)$ for another polynomial $P_{N-1}(z)$ of order $N-1$. But we can repeat this process for $P_{N-1}(z)$ and finally we have $P_N(z) = P_0(z)\prod_{i=1}^N(z-z_i)$, hence it has N roots.

Example Consider $I = \oint_C \bar{z}dz$ on $C : |z| = a$. On the circle, $z = ae^{i\theta}, dz = iae^{i\theta}d\theta$ so $\bar{z} = ae^{-i\theta}$. Hence we have $I = \int_0^{2\pi} ae^{-i\theta}(iae^{i\theta})d\theta = 2\pi a^2 i$.

Example Consider $I = \oint_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{iae^{i\theta}}{a^2e^{2i\theta}} d\theta = 0$. Similarly, $\oint_C \frac{dz}{z^n} = 0, n > 1$. But these are clear form the CIF for derivatives.

Using the CIF to evaluate integrals Consider $I = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^{az}}{z^2(z-2)} dz$. Clearly, the function inside is not analytic at $z = 0, 2$. But $z = 2$ is outside the circle of radius 1. We can hence expand the denominator using partial fractions: $\frac{1}{z^2}\frac{1}{z-2} = \frac{-1}{2z^2} + \frac{-1}{4z} + \frac{1}{4(z-2)}$. Hence we can write the integral as $\frac{1}{2\pi i}\frac{-1}{2} \oint \frac{e^{az}}{z^2} dz + \frac{1}{2\pi i}\frac{-1}{4} \oint \frac{e^{az}}{z} dz + \frac{1}{2\pi i}\frac{1}{4} \oint \frac{e^{az}}{z-2} dz$. But the third term vanishes since $z = 2$ is outside the circle of integration (CIF). Now we let $f(z) = e^{az}$ and $f'(z) = ae^{az}$. Then we appeal to the CIF to rewrite the integrals as $\frac{1}{2\pi i}\frac{-1}{2} f'(0) + \frac{1}{2\pi i}\frac{-1}{4} f(0)$.

Another example Evaluate $I = \int_C \frac{3z-2}{z(z-1)} dz$ for closed curve C that encloses $z = 0$ and $z = 1$, both the singularities of the function in an anticlockwise function. We can write the function in the integral as $\frac{2}{z} + \frac{1}{z-1}$. Since we can deform the contour anyway we want provided it does not cross any singularities, we can re-write the contour as the sum of two integrals that surround each of the singularities: $\int_{C_0} \frac{2}{z} dz + \oint_{C_1} \frac{1}{z-1} dz$ since one of the singularities is outside each of C_1 and C_0.

5.3 29 Oct 2014 Recitation

Theorem for Cauchy-Riemann Equations If the following conditions hold: (1) $f(z)$ is defined in the ϵ neighbourhood of z_0, (2) the first order partial derivatives of u and v exist everywhere in the ϵ neighbourhood of z_0, (3), the first order partial derivatives are continuous at z_0 and satisfy the CR equations, then we have that $f'(z_0)$ exists and f is analytic at z_0.

5.4 31 Oct 2014

Weak Cauchy Assume that $f(z)$ is analytic in D, with $\bar{D} = D + \delta D$, where δD is the boundary of D. Now we note that $f(z)$ may not be analytic at the boundary. The theorem states that $\lim_{C \to \delta D} \oint f(z)dz = \oint_{\delta D} f(z)dz$ provided that $L = \oint_{\delta D} |dz| < \infty$ the length of δD is finite.

Example of Weak Cauchy Consider $f(z) = (z^2 - 1)^{1/2}$ with two branch points. Construct a cut going from -1 to 1 along the real axis. Then $f(z)$ is analytic in the cut plane. Consider the following curves: circle C_1' with radius R, which is large and surrounds the cut, C_1, which is contained in C_1', and δD, which is the curve just surrounding the branch cut. We note that $f(z) = i\sqrt{1-x^2}$ just above the cut and $f(z) = -i\sqrt{1-x^2}$ just below the cut. We consider $\oint_{C_1} f(z)dz = \oint_{C_1 \to \delta D} f(z)dz$ as we deform C_1 towards δD. We can write the RHS as $i\int_1^{-1} \sqrt{1-x^2}dx - i\int_{-1}^{1} \sqrt{1-x^2}dx = -i\pi$ as we move along δD. We now deform C_1 towards C_1', where $|C_1'| = R >> 1$, so we obtain that $(z^2-1)^{1/2} = z - \frac{1}{2z} + O(\frac{1}{R^3})$, and $z = Re^{i\theta}$. Hence $\oint_{C_1'} f(z)dz = \int_{C_1'} zdz - \frac{1}{2}\int_{C_1'} \frac{dz}{z} + O(\frac{1}{R^2})$. The first term will vanish to zero, but the second term is the winding number (times $2\pi i$) for one round around the origin (scaled by $\frac{-1}{2}$), and hence we obtain that the integral is $-i\pi$.

Partial sum Recall that $S_n(z) = a_0 + a_1(z-\alpha) + \ldots + a_n(z-\alpha)^n$, where $\alpha \in \mathbb{C}$ and $a_n \in \mathbb{C}$. Let $f(z)$ and $\{S_n(z)\}$ be given in G of the complex plane. Then we say that $\{S_n(z)\}$ converges uniformly to $f(z)$ in G if $\forall \epsilon > 0$, $\exists N(\epsilon)$ such that $|f(z) - S_n(z)| \leq \epsilon$ for $n \geq N(\epsilon), z \in G$. Write $f(z) = \lim_{n\to\infty} S_n(z)$ uniformly in G, so $f(z) = \sum_{k=0}^{\infty} a_k(z-\alpha)^k$. Note that $N(\epsilon)$ only depends on ϵ, and not on z. If $N(\epsilon, z)$, then the sequence is not uniformly convergent.

Taylor Series Suppose we have some $f(z)$ which is analytic in some open disk $|z - z_0| < R$. Then for each z in the disk, we can write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ and the series is uniformly convergent.

Proof Put $z_0 = 0$ WLOG. Consider a circle that lies inside $|z| < R$, C_1: $|z| < r_1$, $r_1 < R$ but that contains z. Hence z is inside C_1 and the big disk. Now the Cauchy Integral formula gives us $f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)d\zeta}{\zeta - z}$. Now we note that $\frac{1}{1-w} = 1 + w + w^2 + \cdots + w^{N-1} + \frac{w^N}{1-w}$. Now we define $w = \frac{z}{\zeta}, w \neq 1$, then we have that $\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \ldots + \frac{z^{N-1}}{\zeta^N} + \frac{z^N}{\zeta^N(\zeta - z)}$. We now can use the Cauchy Integral Formula to evaluate each of the terms. We note that $\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$ is $\frac{f^{(n)}(0)}{n!}$ using the CIF. Then we have that $f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z)$, where $\rho_N(z) = \frac{1}{2\pi i} z^n \oint_{C_1} \frac{f(\zeta)d\zeta}{\zeta^N(\zeta - z)}$. Suppose $|z| = r$. Then $|\zeta - z| \geq ||\zeta| - |z|| = r_1 - r$. Let $M = \max_{|\zeta|=r_1} |f(\zeta)|$. Then using the bounds on integrals, we have that $|\rho_N(z)| \leq \frac{r^N M}{2\pi(r_1 - r)r_1^N} 2\pi r_1 = \frac{Mr_1}{r_1 - r}(\frac{r}{r_1})^N$. But since $r < r_1$, as $N \to \infty$, $|\rho_N(z)| \to 0$. Hence we obtain the Taylor series: $f(z - z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$.

Remarks on the Taylor series

- If $f(z)$ is analytic within circle C_0 centred on z_0, then the convergence of the Taylor series for all z inside the circle is guaranteed.
- The maximum value for the radius of the circle is the distance of z_0 to the closest singularity.
- If there exists constants $a_n, n = 0, 1, 2, \ldots$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z \in C$, then it must be the Taylor series. The Taylor series is unique.
- Suppose $f(z)$ is entire. Then its Taylor series is defined for all points in the finite z-plane.

Chapter 6

Week 6

6.1 Monday 3 Nov 2014

Maclaurin series Recall $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ converges in D inside $|z| < R$ here R is the distance to the nearest singularity of $f(z)$.

List of Maclaurin series for entire functions

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, |z| < \infty$
- $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, |z| < \infty$
- $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

Note that the cosh function is periodic along the imaginary axis for all z since $\cosh z = \cosh(z + 2\pi i)$. Hence $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(z+2\pi i)^{2n}}{(2n)!}, |z| < \infty$.

Maclaurin series for functions with finite radius of convergence

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1$$
$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1) z^n, |z| < 1$$

Absolute convergence, Uniform convergence of power series Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$. A series is absolutely convergent if $\sum_{n=0}^{\infty} |a_n||z|^n < \infty$ for some z.

Facts about convergence

- All absolutely convergent series are uniformly convergent. But the converse is not true.
- Divergent (i.e. non-convergent) series does not imply that it blows up to infinity.
- If $\sum a_n z^n$ converges when $z = z_1 \neq 0$, then it is absolutely convergent for all $|z| < |z_1|$ on the open disk.
- If $\sum a_n z^n$ diverges for some z_1, $|z_1| = R_2$, then it diverges for all $|z| > R_2$.
- Radius of convergence: The circle $|z| = R$ such that the series is convergent for $|z| < R$ and divergent for $|z| > R$. For $|z| = R$, we do not know; it depends on the series. Interesting case study: $\sum \frac{z^n}{n}$ is convergent for $|z| \leq 1$ and divergent for $|z| > 1$ or $z = 1$. Using deMoirve's theorem, we note that this implies that $\sum \frac{\cos n\theta}{n}$ is convergent except at $\theta = 0$ and $\sum \frac{\sin n\theta}{n}$ is convergent.
- Suppose z_1 is a point inside the circle of convergence for the series $S(z)$, write the circle as $|z| = R$. Then the power series $S(z)$ is uniformly convergent inside $|z| \leq |z_1|$.

Tests for convergence: Sufficient condition for absolute convergence except on the radius of convergence Define $A_n = a_n z^n$. Then $S(z) = \sum A_n$. Then $S(z)$ is absolutely convergent if $\sum |A_n|$ is convergent.

Comparison Test Consider $\sum A_n$. If we can show that $|A_n| < B_n \in \mathbb{R}, B_n > 0$. Also suppose that $\sum B_n < \infty$ is absolutely convergent. Then $\sum A_n$ is absolutely convergent.

D'Alembert Test/Ratio Test A series $S(z) = \sum A_n$ is absolutely convergent if $\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| < 1-\epsilon$, $\epsilon > 0$, for all $n > N$. In practice, we say that a series is absolutely convergent if $\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| < 1$ and diverges if the limit is greater than 1.

Finding the radius of convergence Suppose $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists. Then the series is convergent if $\frac{|z|}{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|} < 1$. Hence the radius of convergence is $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$. This condition is sufficient but not necessary!

Example Consider $a_n = P_n$, a polynomial. Then $\frac{a_n}{a_{n+1}} = \frac{P_n}{P_{n+1}} \to 1$ from below. Hence the radius of convergence is unity. The same holds for rational polynomials $a_n = \frac{P_n}{Q_n}$.

Cauchy's Test Necessary and sufficient conditions for $\{A_n\}_{n=1}^{\infty}$ to converge is that for any $\epsilon > 0, \exists N(\epsilon)$ such that $|A_n - A_m| < \epsilon$ for $n, m > N(\epsilon)$.

6.2 5 Nov 2014 Wednesday

Cauchy' Test for Functions A sequence of functions $\{S_n(z)\}_{n=1}^{\infty}$ converges iff for all $\epsilon > 0, \exists N(\epsilon, z)$ such that $|S_n(z) - S_m(z)| < \epsilon$ whenever $n, m > N(\epsilon, z)$. If $N = N(\epsilon)$, not dependent on position, and it holds for all $z \in D$, then $\{S_n(z)\}$ forms a uniform Cauchy sequence.

Limit Superior Define the limsup of a sequence of real numbers $\{x_n\}$ to be the smallest real number such that for any $\epsilon > 0$, there exists only a finite set such that $x_n > S^+ + \epsilon$. Define $\limsup x_n = S^+$.

Examples of limit superior Consider the sequence $\{(-1)^n\}$. Then $\limsup\{(-1)^n\} = 1$. Consider the sequence $x_n = (-1)^n(1/20 + 1/n)$. Then $\limsup x_n = 1/20$ and $\liminf x_n = -1/20$.

Cauchy's Test using Limsup $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent if $\limsup |a_n z^n|^{1/n} = S^+ < 1$. This means that given $\epsilon > 0$, there exists $N(\epsilon)$ such that $|a_n z^n|^{1/n} \leq S^+ + \epsilon < 1$ for all $n > N(\epsilon)$. Call $\sum a_n z^n$ divergent if $\limsup |a_n z^n|^{1/n} > 1$. Observe that since the series is convergent if $|z| \limsup |a_n|^{1/n} < 1$, then $|z| < \frac{1}{\limsup |a_n|^{1/n}}$. The RHS is the radius of convergence R.

Examples Consider $\sum_{n=0}^{\infty} \frac{2^n}{3^n+4^n} z^n$. Then $a_n^{1/n} = \frac{1}{2}\left[1 + \left(\frac{3}{4}\right)^n\right]^{-1/n} \to e^{-\frac{1}{n}(3/4)^n}\frac{1}{2}$ hence $R = 2$.

Example 2 Consider $\sum_1^{\infty} \sin(n\alpha) z^n, \alpha = a + ib$. Then $|\sin n\alpha|^{1/n} = [\sinh^2(nb) + \sin^2 na]^{1/2n} \to e^{|b|}$. Hence $R = e^{-|b|}$.

Gamma function Define $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$. This is only defined for $\Re(z) > 0$ because $\int_0 t^{x-1} dt$ at $t = 0$ is only convergent for $x > 0$. If z is a positive integer, we have that $\Gamma(n+1) = n!$.

Stirling's Approximation $\Gamma(n+1) = n! \approx \sqrt{2\pi} e^{-n} n^{n+1/2}$ for $n \to \infty$. Hence $(n!)^{1/n}$ goes as n/e. Hence for $\sum_{n=0}^{\infty} n! z^n$ has $R = \frac{1}{\limsup (n!)^{1/n}} = \frac{1}{n/e} \to 0$ as $n \to \infty$. On the other hand, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has $R \to n/e$ which goes to infinity as $n \to \infty$.

Topics for Midterm: Elementary functions, branch points, analytic functions, Integrations

Integration of power series multiplied by function Recall that $S(z) = \sum_{n=0}^{\infty} a_n z^n$ is a continuous function inside its radius of convergence. Let C be a contour inside the circle of convergence. Let $g(z)$ be continuous on C. Then $\sum_C g(z) S(z) dz = \sum a_n \int_C g(z) z^n dz$, the integration can be done term-by-term.

Multiplying by one Consider $g(z) = 1$. Then we have that $\oint_C g(z) z^n dz = \oint_C z^n dz = 0$, for $n = 0, 1, 2, \ldots$ hence $\oint_C S(z) dz = 0$, for all contours C inside the circle of convergence. Then by Morera's theorem, we have that $S(z)$ is analytic in its circle of convergence.

Term-by-term differentiation The power series can be differentiated term-by-term for all z inside the circle of convergence. Hence if $S(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < R$, then $S'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n a_n z^{n-1}, |z| < R$.

Proof of term-by-term differentiation Write $g(\zeta) = \frac{1}{2\pi i(\zeta - z)^2}$. Apply the integration of power series multiplied by a function. Then we have $\oint_C g(\zeta)S(\zeta)d\zeta = \frac{1}{2\pi i}\oint \frac{S(\zeta)d\zeta}{(\zeta - z)^2}$. But by the Cauchy integral formula, the RHS is just $S'(z)$. Applying this to individual powers, we have that $\oint_C g(\zeta)\zeta^n d\zeta = \frac{1}{2\pi i}\oint_C \frac{\zeta^n d\zeta}{(\zeta - z)^2} = \frac{d}{dz}z^n$ by the CIT. Summing all the contributions, we have that $\sum a_n \oint_C g(\zeta)\zeta^n d\zeta = \sum a_n \frac{d}{dz}z^n = S'(z)$, and the identity follows. We can repeat this process for higher derivatives by choosing a different $g(\zeta) = \frac{1}{2\pi i(\zeta - z)^n}$.

Example $\frac{1}{z}$ can be expanded as a Taylor series around $z = 1$ to obtain $\sum_{n=0}^{\infty}(-1)^n(z-1)^n, |z-1| < 1$. Differentiating on each side, we obtain that $\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n$. Hence we now have the Taylor series for $\frac{1}{z^2}$ at $z = 1$.

Caveat While integration and differentiation works term-by-term for power series within its radius of convergence, there are some exceptions. Consider $f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$. This series converges to $\frac{1}{2}x$ for $-\pi < x < \pi$. However, differentiation term-by-term gives a sum that does not converge.

6.3 Friday 7 Nov 2014

Multiplication and Division of Power Series Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, for $|z| < R$, where R is the smaller of the radii of convergence of both series. Then $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. To prove, differentiate each side and use the Maclaurin series. Similarly, $\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n z^n$ whenever $g'(z) \neq 0$.

Zeros of analytic functions Suppose $f(z)$ is analytic at $z = z_0$. Then we say that $f(z)$ has a zero of order m at z_0 if the following conditions hold:

- $f(z_0) = 0$
- $f^{(j)}(z_0) = 0, j = 0, \dots, m-1$
- but $f^{(m)}(z_0) \neq 0$

Near $z = z_0$, we can write $f(z) = (z - z_0)^m[a_m + a_{m+1}(z - z_0) + \dots]$, where $a_m = \frac{f^{(m)}(z_0)}{m!} \neq 0$. We note that the series in the square brackets converges where $f(z)$ converges. Hence we can write $f(z)$ for an mth order zero as $f(z) = (z - z_0)^m g(z)$, where $g(z) \sum_{n=m}^{\infty} a_n(x - x_0)^n$ is analytic at z_0 and $g(z_0) = a_m \neq 0$.

Simple Zero A function with a 1st order zero is said to have a simple zero. That is $f(z_0) = 0$ but $f'(z_0) \neq 0$.

Limit Point of Zeroes Suppose $f(z) = 0$ for $z = z_1, z_2, \dots, z_n, \dots$ (so f has a sequence of zeroes) such that $\exists \alpha$ such that $\lim_{n\to\infty} z_n = \alpha$. Then we call α a limit point/point of accumulation of zeros. Using limit notation, there exists $\epsilon > 0$ such that $|z_n - \alpha| < \epsilon$ for $n > N(\epsilon)$. This means that for all circles around α, the circle will contain an infinite number of zeros. We say that the zeros are not isolated.

Example with limit point Consider $f(z) = \sin(1/z)$, which as zeros at $z_n = \frac{1}{n\pi}, n = 1, 2, \dots$. The limit point of f will be zero, because we can draw a circle around $z = 0$, which will contain an infinite number of zeros. Hence $\alpha = 0$ is a limit point of zeros.

Theorem: Analytic function has isolated zeros only The zeros of an analytic function inside its domain of analyticity are always isolated. There will not be any accumulation points. Exception: $f(z) = 0$ has accumulation points everywhere.

Converse of Theorem If there exists an accumulation point and $f(z)$ is analytic, then $f(z) = 0$.

Corollary Suppose $f(z) = g(z)$ for a series of points $\{z_n\} \in D$ such that $\lim_{n\to\infty} z_n = \alpha \in D$, where D is the domain of analyticity for both f and g. Then $f(z) = g(z)$ everywhere in D. See this by defining $F(z) = f(z) - g(z)$ as another analytic function in D. Then $F(z)$ has an accumulation point at α. Hence, by the converse of the theorem above, it must be that $F(z) = 0$ over D.

Example Consider $f(z) = \sin^2 z + \cos^2 z$. Also consider $g(z) = 1$. Take $z \in \mathbb{R}$. We know that along the real axis $f(x) = g(x)$. Constructing any limit of points along the real axis, and using the theorem above, we can show that $f(z) = g(z)$.

Maximum Modulus Principle Suppose $f(z)$ is analytic in D and continuous in the closed domain $\bar{D} = D + \partial D$. Then either of the two statements are true:

- $f(z)$ is a constant in D.
- $|f(z)| < M$, where M is the maximum value of $|f(z)|$ on ∂D.

$f(z)$ cannot have a maximum modulus in D. The proof is by contradiction. Suppose $|f| = M$, a maximum, at some $z = \alpha \in D$. Then $|f(z)| \leq M$ in some δ neighbourhood surrounding α. But the Cauchy Integral Formula says that $f(\alpha) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-\alpha}dz$. Writing $z = \alpha + \delta e^{i\theta}$, we have that $\frac{1}{z-\alpha} = \frac{1}{\delta e^{i\theta}}$. Hence $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + \delta e^{i\theta})d\theta$. Dividing both sides by $f(\alpha)$, we have that $1 = \frac{1}{2\pi}\int_0^{2\pi} \rho e^{i\phi} d\theta$, where $\rho e^{i\phi} = \frac{f(\alpha+\delta e^{i\theta})}{f(\alpha)}$. Writing out the real and imaginary parts of $\rho e^{i\phi}$, and noting that $\rho \leq 1$ because $f(\alpha)$ is a local maximum, we can show that the only values of ρ and ϕ that satisfy the equation are that $\rho = 1$ and $\phi = 0$. Hence $f(z) = f(\alpha)$ =constant inside the circle. Now we pick another point inside the δ circle, and draw another circle δ_1 around it. Now $f(\alpha)$ is also a maximum in the union of the δ circle and the δ_1 circle. Keep repeating this for every polygonal path connecting two points in the domain to show that $f(z)$ =constant over the whole domain.

Example Suppose $f(z) = 2$ on $|z| = 1$. Inside $|z| = 1$, $f = 2$ is impossible unless $f = 2$ everywhere.

Schwarz Lemma Suppose that the following are true:

- $|f(z)| \leq M$ on $|z| = 1$.
- $f(0) = 0$
- $f(z)$ is analytic inside and on the unit circle $|z| = 1$.

Then either $|f(z)| < M|z|$ for $|z| < 1$ or $f(z) = z\times$complex constant. Prove this by applying the maximum modulus principle to $g(z) = \frac{f(z)}{z}$.

Analytic functions are uniquely determined by its values on the boundary Weaker form: $f(z)$ can be uniquely determined up to a constant by either its real or imaginary parts on its boundary. This is called a boundary value problem.

Singularities of Analytic Functions A singularity is a point where $f(z)$ is not analytic. If $f(z)$ is analytic and single-valued in the punctured disk $0 < |z - \alpha| < R$, but is not analytic at $z = \alpha$. Note that α is not a branch point, because $f(z)$ is single valued in the neighbourhood. Note also that the singularity is isolated, because there is a circle of radius $R_1 < R$ around α such that there is no other singularity other than the one at $z = \alpha$. For example, $\frac{1}{1-z}$ has an isolated singularity at $z = 1$. However, functions with a branch point do not fulfil this condition. Also, $\frac{1}{\sin(1/z)}$ has a denominator with a point of accumulation at $z = 0$. Hence the point $z = 0$ is a non-isolated singularity.

Chapter 7

Week 7

7.1 10 Nov 2014

Behaviour of Isolated Singularities

- **1. Removable singularity**: An IS at $z = \alpha$ is removable if (1) $\lim_{z\to\alpha}(z-\alpha)f(z) = 0$ (2) equivalently, $f(z)$ can be made analytic at $z = \alpha$ by assigning $f(\alpha)$. For example $f(z) = \frac{\sin z}{z}$ is singular at $z = 0$. But if we define $f(0) = 1$, then the function becomes analytic at $z = 0$.

- **2. Pole**: An IS is a pole of order m if $f(z)$ near the singularity can be written as $f(z) = \frac{g(z)}{(z-\alpha)^m}$, where $m \geq 1, m \in \mathbb{Z}$, and $g(z)$ analytic at $z = \alpha$, and $g(\alpha) \neq 0$. If $m = 1$, then the pole is said to be simple. Note that we can write the condition as $\lim_{z\to\alpha}(z-\alpha)^m f(z)$ =constant.

- **Necessary and sufficient condition for an IS to be a pole**: $\lim_{z\to\alpha}|f(z)| = \infty, \forall$ paths $z \to \alpha$. Equivalently, given any $M > 0$, there exists $\delta(M) > 0$ such that $|f(z)| > M$ whenever $0 < |z-\alpha| < \delta(M)$.

- **Poles and Zeroes** If f has a pole of order m at $z = \alpha$, then $\frac{1}{f}$ has a zero of order m at $z = \alpha$.

- **3. Essential Singularity** Example: $f(z) = e^{1/z}$. Examine this function near $z = 0$. Separating into real and imaginary parts, we obtain that $f(z) = e^{[x/(x^2+y^2)]}[\cos(y/(x^2+y^2)) - i\sin(y/(x^2+y^2))]$. Consider $z = x, x > 0, y = 0$. Then $f(z) = e^{\frac{1}{x}} \to \infty$ as $x \to 0+$. Also, $\lim_{z\to\ 0} x^m e^{1/x} \to \infty$ for all m. Now consider $z = x, x < 0, y = 0$. Now $f(x) \to 0, x \to 0-$. If $z = iy, x = 0$, then $f(z) = [\cos(1/y) - i\sin(1/y)]$ which has an accumulation of zeros at $z = 0$.

Picard's Theorem Let $f(z)$ have an isolated essential singularity at $z = \alpha$. Then in the neighbourhood of $z = \alpha$, $f(z)$ assumes all possible complex values an infinite number of times except for possibly one value. In other words, for all $A \in \mathbb{C}$, the equation $f(z) - A = 0$ has an infinite number of roots inside $|z-\alpha| = \delta$, for all $\delta > 0$, except possibly one A.

Example of Picard's Theorem Consider $f(z) = e^{1/z}$. Then $e^{1/z} + 1 = 0$ has an infinite number of roots $z = \frac{1}{(2n+1)\pi i}, n = 0, \pm1, \pm2, \ldots$. But there is are no roots for $e^{1/z} = 0$. This is the exception in Picard's theorem.

Non-isolated Singularities

- If a function f has a limit point of zeros, then we will have an isolated essential singularity there. Example: $\sin(1/z)$.

- If a function has an accumulation of poles, then there will be a non-isolated essential singularity there. E.g. $\frac{1}{\sin(1/z)}$.

Point at infinity Let $z = \frac{1}{\zeta}$ and consider $f(\frac{1}{\zeta})$ when $\zeta = 0$. Then a singularity at infinity corresponds to the singularities of $f(1/\zeta)$ at $\zeta = 0$. Generally, when $|f| \to \infty$ as $|z| \to \infty$ for all $\arg z$, then there will be a pole at $z = \infty$. Then we can write $f(z) = z^m g(z)$ near infinity, where $|g(z)| < M$ is bounded as $z \to \infty$. **Example 1** Let $P_N = z^N + a_1 z^{N-1} + \ldots$. Then $P_N(1/\zeta) \to \frac{1}{\zeta^N}$ as $\zeta \to 0$ and P_N has a pole of order N at $z = \infty$. **Example 2** Put $f(z) = e^z$. Then $f(1/\zeta) = e^{1/\zeta}$, which has an isolated essential singularity at $\zeta = 0$. Hence e^z has an isolated essential singularity at infinity. **Example 3** Take $f(z) = \frac{(z^2-1)(z-2)^3}{(\sin \pi z)^3}$. Examine the denominator $(\sin\pi z)^3$. It has zeros for integer values of z. We can write $\sin(\pi z) = (-1)^n(z-n) + \ldots$ so $(\sin(\pi z))^3 = (-1)^{3n}(z-n)^3 + \ldots$. The numerator of the original function has a zero of order 1 at $z = \pm1$ and a zero of order 3 at $z = 2$. Hence the function has a pole or order 2 at $z = \pm1$. We also have that at $\lim_{z\to 2}\frac{z-2}{\sin(\pi z)} = \frac{1}{\pi}$ and $\lim_{z\to2} f(z) = \frac{3}{\pi^3}$ so $f(z)$ has a removable singularity at $z = 2$. For all other values of $z = n, n \in \mathbb{Z}$, $f(z)$ has a pole of order 3. Now consider the point at infinity. Then $f(1/\zeta) = (\frac{1}{\zeta^2} - 1)(\frac{1}{\zeta} - 2)^3 \frac{1}{\sin^3(\pi/\zeta)}$. The first two terms give a pole of order 5 at $\zeta = 0$, but the third term will give a non-isolated singularity at $\zeta = 0$.

Laurent Series Consider $f(z)$ analytic and single-valued in some domain D, which is the finite annulus between $R_1 < |z-\alpha| < R_2 \leq \infty$. Then we can write:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-\alpha)^n}$$

where the first term is absolutely convergent in the circle $|z-\alpha| = R_2$ and the second term is absolutely convergent OUTSIDE the circle $|z-\alpha| = R_1$. The second term is called the singular part. If all $b_n = 0$, there is no singular part and the Laurent series is just the Taylor series. The coefficients a_n and b_n can be written as integrals (WLOG $\alpha = 0$):

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z)^{n+1}}$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z)^{-n+1}}$$

7.2 12 Nov 2014 Wednesday

Laurent Series Proof Let C_1 be the curve that traces out $|z| = R_1$ clockwise, and C_2 be the curve that traces out $|z| = R_2$ anticlockwise. Consider a point in the annulus. Then we can write $f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)d\zeta}{\zeta - z}$. We note that for $|z| < R, \zeta$ on R_2, we can expand $\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \ldots\right)$. Similarly, for $|z| > R_1$, ζ on C_1, we have $\frac{1}{\zeta - z} = -\frac{1}{z}(1 + \frac{\zeta}{z} + \frac{\zeta^2}{z^2} + \ldots)$. Substitute these expressions into the CIF equation to obtain the Laurent series.

Laurent series around isolated singularity Consider an isolated singularity at $z = \alpha$. Define $R_1 = 0$. Then $f(z)$ is analytic on $0 < |z-\alpha| < R$, where R is the distance to the nearest other singularity. We can combine the summation by reindexing $m = -n$. Then we have $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n, a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{(\zeta-\alpha)^{n+1}}$ for C being any circle $|z-\alpha| =$constant, such that $0 < const < R$.

Remarks on the Laurent series

- If $f(z)$ is analytic on the punctured disc $0 < |z-\alpha| < R$, then the Laurent series is unique and converges to $f(z)$.
- If $a_n = 0$ for all $n \leq 0$, then the singularity is removable.
- If $a_n = 0$ for all $n < -m$, then $f(z)$ has a pole of order m at $z = \alpha$.
- If the set of non-zero $a_n, n < 0$ is infinite, then $z = \alpha$ is an essential singularity.
- The radii R_1 and R_2 are determined by the singularities of f. The Laurent series cannot converge outside the annulus defined by R_1 and R_2.

Residues The coefficient of $n = -1$, which can be written as $a_{-1} = \frac{1}{2\pi i} \oint f(\zeta)d\zeta$ is called the residue.

Calculation of residues

- Suppose $f(z)$ has a simple pole at $z = \alpha$. Then we know that $a_{-1} = \lim_{z\to\alpha}(z-\alpha)f(z)$.
- Suppose $f(z) = \frac{g(z)}{h(z)}$, where $g(\alpha)$ is finite and $h(z)$ has a simple zero at $z = \alpha$. Then $a_{-1} = \frac{g(\alpha)}{h'(\alpha)}$. Note that $\lim_{z\to\alpha} \frac{(z-\alpha)g(z)}{h(z)}$ has both numerator and denominator go to zero. Then using L'Hopital's rule, we have that $a_{-1} = \lim_{z\to\alpha} \frac{g(\alpha)}{h'(\alpha)}$.
- If a function has a pole of order m at $z = \alpha$. Then we can write $f(z) = (z-\alpha)^{-m}[a_{-m} + \ldots + a_{-1}(z-\alpha)^{m-1} + a_0(z-\alpha)^m + \ldots]$. Then we note that a_{-1} is the $m-1$st derivative of the function in the brackets (up to multiplication by a factorial): $a_{-1} = \frac{d^{m-1}}{dz^{m-1}} \frac{(z-\alpha)^m f(z)}{(m-1)!}|_{z=\alpha}$.
- If $f(z)$ has an essential singularity, then the residue there is generally hard to find. Perform the full Laurent series.

Example Define $f(z) = \frac{1}{(z-1)z}$ which has simple poles at $z = 0, z = 1$. Consider the Laurent series about $z = 0$. Define the interior of the unit circle Region I and the exterior Region II. When $0 < |z| < 1$, then we have $R_1 = 0, R_2 = 1$. When $1 < |z| < \infty$, $R - 1 = 1, R_2 = \infty$. There are distinct Laurent series in the two regions. We can calculate the Laurent series using two different methods.

Method 1: In Region I, consider $a_n = \frac{1}{2\pi i} \oint_{C_I} \frac{d\zeta}{\zeta^{n+2}(\zeta-1)}, C_I$ is in $|z| < 1$. Then we can use the Cauchy Integral formula to obtain that $a_n = \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left(\frac{1}{\zeta - 1} \right)_{\zeta=0} = \begin{cases} -1, & 1+n \geq 0 \\ 0, & n < -1 \end{cases}$. Hence $f(z) = -\frac{1}{z} - 1 - z^2 - z^3 + \ldots$ and hence we have the residue at $z = 0$ is -1. In Region II, consider $a_n = \frac{1}{2\pi i} \oint_{C_{II}} \frac{d\zeta}{\zeta^{n+2}(\zeta-1)}$, where C_{II} is a circle outside $|z| = 1$. Performing the integration, we will obtain $f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \ldots$ so the residue at $z = 0$ is 0.

Method 2: Do a partial fraction expansion $f(z) = \frac{1}{z-1} - \frac{1}{z}$. Then just perform a binomial expansion of the first term in either region.

Note that we do not have to take the Laurent series around the singularities. We can perform the expansion around any point α. But now we need to draw circles around α that touch each of the singularities, and the Laurent series will take on a different form in each of the three regions.

Example Let $f(z) = Log(1+z)$, which has a branch point at $z = -1$. Construct a branch cut along the negative real axis going to infinity. Construct the Laurent series around $z = 0$. Then we will have that $Log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots, |z| < 1$. Note that for $|z| > 1$, then $f(z)$ is no longer single-valued in the annulus and hence the LE does not exist.

7.3 14 Nov 2014 Friday

Residue Theorem Consider $f(z)$ analytic in D, except at a finite number of isolated singularities. Index the singularities by $z = \alpha_j$, and let the residues at each singularity be $r_j = a_{-1}^{(j)}$ for the Laurent series about each of the isolated singular points. Statement of theorem: $\oint_C f(z)dz = 2\pi i \sum_j r_j, \forall C \in D$, where the sum is over all α_j in C.

Proof of residue theorem We can deform the contour surrounding the singularities to consider the arbitrarily small contours surrounding each of the singularities. Then we have that $\oint_C f(z)dz = \sum_j \oint_{C_j} f(z)dz$, where $C_j : |z - \alpha_j| = \delta_j$ is a circle around the jth singularity with radius δ_j. Then for each $\oint_{C_j} f(z)dz = \oint_{C_j} \left[\sum_{-\infty}^{\infty} a_n^{(j)} (z - \alpha_j)^n \right] dz$ after writing the Laurent series around each singular point. But by the Cauchy integral formula, the integral of each of the powers of z on the curve will vanish except the integral involving the power of negative one. Hence we have that $\oint_{C_j} = 2\pi i a_{-1}^{(j)} = 2\pi i r_j$. Hence the total contour integral is $2\pi i \sum_j r_j$ for all singularities within the contour C.

Residue of a pole $r_j = a_{-1}^{(j)} = \frac{d^{m-1}}{dz^{m-1}} \frac{(z-\alpha_j)^m f(z)}{(m-1)!}$.

Evaluation of definite integrals - General Consider $I = \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$ on some circle. Consider the unit circle. Then $\cos\theta = \frac{1}{2}(z + \frac{1}{z}), \sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ since $e^{i\theta}$. Hence we can write the integral as $I = \int_0^{2\pi} F(\frac{1}{2}(z + 1/z), \frac{1}{2i}(z - 1/z)) \frac{dz}{iz}$ because $dz = ie^{i\theta}d\theta \implies d\theta = \frac{dz}{iz}$.

Evaluation of definite integrals - Example $I = \int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, 0 < a < 1$. Considering the unit circle, we can use the notation of the previous section to write $I = \oint_{|z|=1} \frac{dz}{iz(1 + \frac{a}{2i}(z - 1/z))} = \frac{2}{a} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2ia}{z} - 1}$. Hence we just need to find the residues at the poles of $\frac{1}{z^2 + 2ia/z - 1}$. The singularities of the function inside the integral is located at $z = \frac{i}{a}\left(\pm\sqrt{1 - a^2} - 1\right)$. Note that there will be one root inside the unit circle and another outside. The singularity within the unit circle is located at $\frac{i}{a}(\sqrt{1 - a^2} - 1)$. We write $f(z) = \frac{2}{a} \frac{1}{z^2 + 2ia/z - 1} = \frac{2}{a} \frac{1}{(z - z_1)(z - z_2)}$ so the residue at the inner singularity within the unit circle is $a_{-1}^{(1)} = \lim_{z \to z_1} (z - z_1)f(z) = \frac{1}{i\sqrt{1 - a^2}}$.

Notes on definite integrals

- If we replace $\theta \to \theta + \pi/2$ then we can evaluate functions with $\cos\theta$ instead of $\sin\theta$.
- If we have higher order trigonometric terms in the denominator (like $\sin^2\theta$), then we can write it in terms of a linear combination of first-order sines and cosines to make the polynomial in z (when substituting in the sine and cosine in terms of z) simpler and lower order.

Example Consider $I = \int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta$. We can write this as the real part of $\Re\left(\int_0^{2\pi} e^{\cos\theta - i(n\theta - \sin\theta)} d\theta\right)$ note the negative sign does not matter because cosine is even. But we can write this integral on the unit circle to be $\Re\left(\oint_{|z|=1} e^z \frac{1}{z^n}\frac{dz}{iz}\right)$. This is because $e^{\cos\theta + i\sin\theta} = e^z$ and $e^{-inz} = \frac{1}{z^n}$. But now we have that we want to find $\Re\left(\frac{1}{i}\oint_{|z|=1}\frac{e^z}{z^{n+1}}dz\right)$, which we can use the Cauchy integral formula to write $I = \Re\left(\frac{2\pi i}{in!}\right) = \frac{2\pi}{n!}$.

Summing series using contour integration Consider the useful function $\pi\cot(\pi z)$, which has simple zero at $z_n = n, n \in \mathbb{Z}$. The residue at each of the poles is $r_n = 1, \forall n$. Furthermore, this function will be bounded on a well-chosen contour.

Summing series using the useful function Consider the series $\sum_{n=1}^{\infty}\frac{1}{n^4+\alpha^4}$. Now define $P_n = \frac{1}{2\pi i}\oint_{C_n}\frac{\pi\cot(\pi z)}{z^4+\alpha^4}dz$. Now consider the square C_N centred at the origin and having the points $z = -N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N$ in its interior. Let the length of the square have value $2N+1$. Now the contour integral on the square is going to be the sum of the residues at all of the interior points. But the residue of $\pi\cot(\pi z)$ is 1, hence the residue of the function in the integral is $\frac{\pi}{n^4+\alpha^4}$ at each point $n = -N, \ldots, N$. Now the denominator also has zeroes at $z_j = \alpha e^{i\pi/4}, \ldots, \alpha e^{7i\pi/4}$, which will contribute another 4 residues to the function. The residue at each of these additional singularities is $r_j = \frac{\pi\cot\pi z_j}{4z_j^3}$. Hence $P_N = \left[\sum_{-N}^{N}\frac{\pi}{n^4+\alpha^4} + \sum_{j=1}^{4}\frac{\pi\cot\pi z_j}{4z_j^3}\right]\frac{1}{2\pi i}$. On the square, $\cot(\pi z)$ is bounded. Hence P_N will be of order $\frac{4(2N+1)}{N^4}\cot(N+\frac{1}{2})$, which goes as $\frac{1}{N^3}$ and hence goes to zero as $N \to \infty$.

General Series Summation Consider $S = \sum_n f(n)$. Define the integral $\frac{1}{2\pi i}\oint_{C_N} f(z)\pi\cot(\pi z)dz$.

Chapter 8

Week 8

8.1 Monday 17 Nov 2014

Integrals over the Real Axis Usual requirements:

- $f(x) = \lim_{y\to 0} f(z)$ represents some part of $f(z)$ on the real axis.
- $f(z) \to 0$ as $z \to \infty$ in either the upper or lower half-complex-plane.
- $f(z)$ has one or more singularities (otherwise contour integration will be zero).

Improper integrals Write $\int_{-\infty}^{\infty} f(x)dx$ is the limit: $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$. We require that the limit exist to evaluate the improper integral. Typically, for the integral to exist, we can check that $f(x)$ is monotonically decreasing and ensure that $f(x)$ goes as $x^{-1+\nu}, \nu > 0$ as $x \to \infty$. The case when $f(x)$ goes as x^{-1} will not allow $f(x)$ to converge.

Picking a contour Pick the semi-circular contour with the base along the real axis and centred at the origin. Call this contour C_R, which we split into two parts: $\oint_{C_R} = \int_{-R}^{R} dx + \int_{semicircle}$. This value is going to be $2\pi i \sum_j r_j$.

Example If $|f|$ goes as $O(\frac{1}{|z|^2}$ at infinity, then the integral over the big semicircle as $R \to \infty$ will look like $\int O(\frac{1}{R^2}) iRe^{i\theta} d\theta$ which goes like $\frac{1}{R}$, and hence goes to zero.

Example Consider $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ Let $f(z) = \frac{1}{1+z^4}$. Clearly, when z is large, $|f(z)$ goes as $\frac{1}{|z|^4}$ and the integral of $f(z)$ along the semi-circular part will go as $O(\frac{1}{R^3})$ and hence go to zero as $R \to \infty$. The poles of $f(z)$ occur when $z^4 = -1$, which we can write as $z = e^{i\pi/4}, e^{3i\pi/4}, e^{-i\pi/4}, e^{-3i\pi/4}$. Of these 4 roots, two will be in the upper half plane and two will be in the lower half plane. Hence we just require the residues of the singularities in the one half-plane. Observe that $r_j = \frac{1}{4z_j^3}$. In the upper half plane, we find that $r_1 = \frac{1}{4}e^{-3i\pi/4}, r_2 = \frac{1}{4}e^{-9i\pi/4}$. Hence we have that:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} + 0 = 2\pi i(r_1 + r_2) = \frac{\pi}{\sqrt{2}}$$

Notes

- Note that we chose the upper half-plane in the example immediately above. This is called closing in the upper half plane. We could have closed in the lower half plane as well because $f(z)$ in that case also vanishes as $z \to \infty$ in both the half-planes. However, if we had closed in the lower half-plane, then the contour is negatively oriented and hence we need to introduce an additional minus sign.
- Note that in the previous example, $f(-x) = f(x)$ is an even function. Hence $\int_0^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$.

Harder Example Define $F(a) = \int_0^{\infty} \frac{\cos ax}{1+x^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}dx$. Set $a > 0, a \in \mathbb{R}$. We could try $f(z) = \frac{\cos az}{1+z^2}$. But this is not going to work because $f(z)$ will go like $\cosh ay$ when $y \to \pm\infty$ and hence $f(z)$ becomes unbounded there. The underlying problem is that there is an essential singularity at infinity for $\cos az$. We need to try something different. Note that $\cos ax = \Re(e^{iax})$. So we try $\int_{-\infty}^{\infty} \frac{e^{iax}}{1+z^2}dx$ and take the real part (which works). Take $f(z) = \frac{e^{iaz}}{1+z^2} = \frac{\cos az + i\sin az}{1+z^2}$. Hence the poles of $f(z)$ are at $\pm i$ and $r(z = i) = \frac{e^{-a}}{2i}$. We close in the upper half-plane. Then we write:

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iax}}{1+x^2}dx+\int_{\text{upper semicircle}}\frac{e^{iaz}}{1+z^2}dz=\pi e^{-a}$$

We examine the behaviour of the function on the semicircular portion. Since $a > 0$, we have $iaz = iax - ay$ so $e^{iaz} = e^{-ay}e^{iax}$ which goes to zero when $y \to \infty$. Also, $\frac{1}{1+z^2}$ goes as $O(1/R^2)$ as $R \to \infty$ hence the integral on the upper half circle will go to zero. Hence taking the limit as $R \to \infty$, we obtain that

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{1+x^2}dx=\pi e^{-a}$$

$$\implies \int_{0}^{\infty}\frac{e^{iax}}{1+x^2}dx=\pi e^{-a}\frac{1}{2}$$

Note that we cannot close in the lower-half plane because $a > 0$ and e^{-ay} will blow up in the lower half plane when $y < 0$. However, if $a < 0$, then e^{-ay} will go to zero in the lower half plane and we should close in the lower half plane instead of the upper half plane. We will obtain $F(a) = \frac{\pi}{2}e^a, a < 0$.

Example Consider $I = \int_{-\infty}^{\infty}\frac{\sin x}{x}dx$. Note that $\sin x/x$ is an even function and has a removable singularity at $x = 0$. Note that we cannot use $\oint \frac{\sin z}{z}dz$ because $\sin z$ has an essential singularity at infinity and hence the integral along the upper semicircle will blow up. Instead, we try $\oint \frac{e^{iz}}{z}dz$. There exists a simple pole at $z = 0$. We would like to close in the upper half-plane but the contour would pass through the origin. Hence we need to use an indented contour which makes a small semicircle of radius ϵ around the origin. There are now four sections of the contour:

$$\oint\frac{e^{iz}}{z}dz=\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}dx+\int_{\epsilon}^{R}\frac{e^{ix}}{x}dx+\int_{|z|=\epsilon,upper,CW}\frac{e^{iz}}{z}dz+\int_{|z|=R,upper,ACW}\frac{e^{iz}}{z}dz=0$$

Note that the first two terms are equal and can be written as $2i\int_0^R\frac{\sin x}{x}dx = i\int_{-R}^{R}\frac{\sin x}{x}dx$. For the third term, we write $z = \epsilon e^{i\theta}$ so $dz = i\epsilon e^{i\theta}d\theta$. Hence the third term can be written as $\int_\pi^0(e^{i\epsilon e^{i\theta}})id\theta = -i\pi$. For the 4th term, we need to consider Jordan's Lemma.

Jordan's Lemma Consider $\int_{|z|=R}|e^{iz}|\cdot|dz|$. This is the same as $R\int_0^\pi e^{-R\sin\theta}d\theta$ because $e^{iz} = e^{-y}(\cos x + i\sin x)$ and hence $|e^{iz}| = e^{-y} = e^{-R\sin\theta}$. Also, $|dz| = Rd\theta$. Take $0 \le \theta \le \pi/2$. Note that with $\theta \in [0, \pi/2]$, $\sin\theta \ge \frac{2\theta}{\pi}$ (can be seen graphically if plotted against θ). Then we have a bound on the integral $\int_{|z|=R}|e^{iz}|\cdot|dz| < 2R\int_0^{\pi/2}e^{-2R\theta/\pi}d\theta$ where we have split the domain into $[0, \pi/2], [\pi/2, \pi]$ and noted that $\sin\theta$ is symmetric about the line $\theta = \pi/2$. The RHS integrates to $2R\frac{\pi}{2R}(1 - e^{-R})$, which goes to π as $R \to \infty$.

Hence we have another bound:

$$\left|\int_{|z|=R}\frac{e^{iz}}{z}dz\right|\le\int\frac{|e^{iz}||dz|}{|z|}=\frac{1}{R}\int|e^{iz}||dz|$$

and the right-most term goes to zero as $R \to \infty$.

Going back to the example, we have that the 4th term goes to zero, so $\int_0^\infty\frac{\sin x}{x}dx = \frac{\pi}{2}$.

8.2 19 Nov 2014 Wednesday

Alternative solution for sin x/x Consider the contour that goes around the pole such that the pole is contained within the interior of the contour. Then we have to use the residue at $z = 0$, which is 1. Rewriting, we have

$$\oint\frac{e^{iz}}{z}dz=\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}dx+\int_{\epsilon}^{R}\frac{e^{ix}}{x}dx+\int_{|z|=\epsilon,lower,ACW}\frac{e^{iz}}{z}dz+\int_{|z|=R,upper,ACW}\frac{e^{iz}}{z}dz=2\pi i$$

Repeating the calculations for the 1st, 2nd and 4th term, and noting that the third term is now in the opposite direction, we will obtain the same solution.

Corollary to Jordan's Lemma Consider $f(z) = \frac{p(z)}{q(z)}$ with $p(z)$ having no zeroes and $q(z)$ having a finite number of zeros. Let all the singularities of f be below $|z| = R$ in the upper half-plane. Also assume that $|f| \leq M_R$ for $|z| = R, y > 0$, and that $\lim_{R\to\infty} M_R = 0$. **Statement** $\lim_{R\to\infty} \int f(z)e^{iaz}dz = 0$ on the upper semicircle at $|z| = R$. to verify exact statement

Trigonometric integrals Note that integrals of the form $\int_0^\infty f(x)\sin x dx$ or $\int_0^\infty f(x)\cos x dx$ exist even though $\lim_{R\to\infty} \int_0^R f(x)dx = \infty$. Requirement: $f(x)$ is monotonically decreasing.

Existence of trigonometric integrals We use integration by parts. Example: $\int_0^R \frac{\sin x}{x}dx = \frac{-\cos R}{R} - \int_0^R \frac{\sin x}{x^2}dx$. The first term goes to zero as $R \to 0$, and the second term converges to zero by the comparison test since $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$ and $\int_0^R \frac{dx}{x^2}$ converges for $R \to \infty$.

Exceptions: When the semicircular integration fails Example: Consider $I(a) = \int_{-\infty}^\infty \frac{e^{ax}}{e^x+1}dx, 0 < a < 1$. Pick $f(z) = \frac{e^{az}}{e^z+1}$. But we note that the denominator has an infinity of zeros $z = (2n+1)\pi i$ and hence the function has an infinity of poles. We need to use another trick that works when the denominator is a periodic function of x. We choose a rectangle defined by $x \in [-R, R]$ and $y \in [0, 2i\pi]$, where the vertical cut is in between the poles $z = i\pi$ and $z = 3i\pi$. Hence there is only one pole in the interior, and it has residue $-e^{i\pi a}$. Then we have that the integral along the bounding rectangle is $\oint_C f(z)dz = 2\pi i(-e^{i\pi a})$. This is going to be equal to the sum of 4 integrals, one along each side of the rectangle. Label the corners of the rectangle as ABCD, where A is at the lower left hand corner and we proceed anticlockwise.

Then we have that $\oint_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$. By the bounds of integrals, we note that along the vertical path BC, $|\int_{BC}|$ is less than $2\pi\frac{e^{aR}}{e^R+1}$, where the part in the fraction is the maximum value of $f(z)$ on the path and 2π is the length of the path. Similarly, along vertical path $|\int_{AD}| \leq 2\pi\frac{e^{-aR}}{e^{-R}+1}$. Hence the values of the integral along the vertical paths vanish as $R \to \infty$. Now we also have $\int_{AB} f(z)dz + \int_{CD} f(z)dz = \int_{-R}^R \frac{e^{ax}}{e^x+1}dx + \int_R^{-R} \frac{e^{ax}}{e^x+1}e^{2\pi ia}dx$. where the extra exponential term comes from us noting that the function is periodic in the imaginary axis with period $2\pi i$. Taking the limit as $R \to \infty$, we have that $\left(1 - e^{2\pi ia}\right) \int_{-\infty}^\infty \frac{e^{ax}}{e^x+1}dx = -2\pi ie^{i\pi a}$ Hence we have that $I(a) = \frac{\pi}{\sin \pi a}$.

Analytic continuation of the previous example Note that in the previous example, we restricted $0 < a < 1$. Now we want to consider the case for any value of a. Note that $I(a)$ has poles at the positions $a = n, n \in \mathbb{Z}$. We can define $I(a)$ to be an analytic function of a, and since we know its value matches that obtained value in the previous example, then by analytic continuation, we can say that this function is the actual analytic function over all values of a.

More identities:

- $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{2}}$
- $\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}}$

Cauchy Principal value integral Consider $\int_a^b f(x)dx$, where $f(x)$ is continuous except at one point. Example $f(x) = \frac{1}{\sqrt{x}}$. If $\lim_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)dx$ exists, then we say that the integral exists as an improper integral. If $f(x)$ is continuous except at $x = c$, and c is within the limits of integration, then we examine $\lim_{\epsilon_1\to 0, \epsilon_2\to 0} \left[\int_a^{c-\epsilon_1} f(x)dx + \int_{c+\epsilon_2}^b f(x)dx\right]$ exists, then we say that the integral $\int_a^b f(x)dx$ exists as an improper integral.

Example $\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} \left[2\sqrt{x}\right]_\epsilon^1 = 2$.

Example Consider $\int_{-1}^1 \frac{dx}{x} = \lim_{\epsilon_1\to 0, \epsilon_2\to 0} \ln \frac{\epsilon_1}{\epsilon_2}$ and hence the limit does not exist if we have the relation $\epsilon_1 \neq \epsilon_2$. But if we have $\epsilon_1 = \epsilon_2$, then we have that the limit exist and is zero. Integrals that exist in such a manner is called the Cauchy Principal Value Integral.

Formal Definition of the CPVI $P \int_a^b f(x)dx = \lim_{\epsilon\to 0} \left[\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^b f(x)dx\right]$.

Example Note that $P \int_{-1}^1 \frac{dx}{x^2} = \infty$ because there is a double pole at $x = 0$.

Example $P \int_{-\infty}^\infty \frac{\cos x}{a^2-x^2}, a \in \mathbb{R}$. Consider $\oint \frac{e^{iz}}{a^2-z^2}dz$ which has poles at $z = \pm a$. Pick the anticlockwise contour on the upper semicircle of large radius R that goes around the poles in the clockwise sense (i.e. interior of contour does not contain the singularities). We now have 6 pieces to consider. We hence have:

$$0 = \oint f(z)dz = \int_{-R}^{-a-\epsilon} f(z)dz + \int_{-a+\epsilon}^{a-\delta} f(z)dz + \int_{a+\delta}^{R} f(z)dz + \int_{|z+a|=\epsilon,\text{around z=-a, upper, CW}} f(z)dz$$
$$+ \int_{|z-a|=\delta,\text{around z=-a, upper, CW}} f(z)dz + \int_{|z|=R} f(z)dz$$

but we can write the first three terms as the Cauchy principal value integral $P\int_{-R}^{R} f(z)dz$, the 4th term is $-\frac{1}{2}(2\pi i\text{Res}(z = -a))$ and the 5th term is $-\frac{1}{2}(2\pi i\text{Res}(z = a))$. The 4th and 5th terms are called half-poles because we are going around the poles halfway. The minus sign comes from the clockwise orientation of these half-poles.

Also, by the Jordan Lemma, the 6th term is on order $O(1/R)$, and hence goes to zero as $R \to \infty$. Hence we write:

$$\oint \frac{e^{iz}}{a^2 - z^2} dz = P\int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} dx - \frac{1}{2}(2\pi i r_{z=-a}) - \frac{1}{2}(2\pi i r_{z=1})$$

8.3 Recitation 19 Nov 2014

Example Consider:

$$w(z) = \frac{1}{z-3} + \frac{1}{z-4}$$

Take the Laurent expansion around $z = 1$ and $2 < |z - 1| < 3$. Write

$$w(z) = \frac{1}{(z-1)-2} + \frac{1}{(z-1)-3}$$
$$\implies w(z) = \frac{1/(z-1)}{1-2/(z-1)} + \frac{-1/3}{1-(z-1)/3}$$
$$\implies w(z) = \frac{1}{z-1}\sum_{n=0}^{\infty}(2/(z-1))^n + \frac{-1}{3}\sum_{n=0}^{\infty}((z-1)/3)^n$$

Take the Laurent expansion around $z = 0$ for $3 < |z| < 4$. Then we have:

$$w(z) = \frac{1/z}{1-3/z} + \frac{-1/4}{1-z/4}$$
$$\implies w(z) = \frac{1}{z}\sum_{n=0}^{\infty}(3/z)^n + \frac{-1}{4}\sum_{n=0}^{\infty}(z/4)^n$$

Take the Laurent expansion around $z = 0$ for $|z| < 3$. Then we have:

$$w(z) = \frac{-1/3}{1-z/3} + \frac{-1/4}{1-z/4}$$
$$\implies w(z) = \frac{-1}{3}\sum_{n=0}^{\infty}(z/3)^n + \frac{-1}{4}\sum_{n=0}^{\infty}(z/4)^n$$

Example 2 Find the first three terms of:

$$\log(\frac{z+1}{z-1})$$

First define the branch. Note the branch points at $z = \pm 1$. Since we want the series for $|z| > 1$, we perform the branch cut in between the branch points.

8.4 Friday 21 Nov 2014

Integration around an arc Consider $f(z)$ that has a simple pole at $z = \alpha$ with residue a_{-1}. We want to integrate around a circular positively oriented arc centred at $z = \alpha$ subtending an angle ϕ and with radius δ. We hence consider:

$$\int_{\delta \to 0, c_\phi} f(z)dz$$

Because $f(z)$ has a simple pole at α, we can write $f(z) = g(z) + a_{-1}(z-\alpha)^{-1}$ with $g(z)$ analytic at $z = \alpha$. Then we can break the integral into two parts to give:

$$\int_{C_\phi} g(z)dz + \int_{C_\phi} \frac{dz}{z-\alpha}$$

We bound the first integral by defining $M = \max|g(z)|$ and note that the length of the arc is $\delta\phi$ so the first integral is bounded by $M\delta\phi$ which goes to zero as $\delta \to 0$. For the second integral, we write $z - \alpha = \delta e^{i\theta}$, $\frac{1}{z-\alpha} = \frac{1}{\delta}e^{-i\theta}, dz = i\delta e^{i\theta}d\theta$. We now explicitly perform the second integral:

$$\int \frac{dz}{z-\alpha} = \int_{\theta_0}^{\theta_0+\phi} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}} d\theta = i\phi$$

Hence we have that $\int_{\delta\to 0, C_\phi} f(z)dz = ia_{-1}\phi$. When we integrate on the full circle, we set $\phi = 2\pi$ so we get consistency with the previous definition of the residue. For a half-pole, we set $\phi = \pi$, and hence it is exactly half the value of the residue.

Note that the previous analysis only holds when we have a pole, not a branch point.

Principal value at infinity Define $P\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$. Note that we can replace $R = \frac{1}{\epsilon}$ and take the limit as $\epsilon \to 0$.

Example: Convergence at infinity Take $P\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{xdx}{x^2+1} = \left[\frac{1}{2}\ln(1+x^2)\right]_{-R}^{R} \to 0$. Hence the Cauchy Principal value exists. However, if we take $\lim_{R\to\infty} \int_0^R \frac{xdx}{x^2+1}$, we note that this latter integral will not exist! It happens that in the former case, the integration across the whole real line has parts that cancel.

Plemelj Formula Consider an arc C going from a to b and consider the function $I(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$. This defines an analytic function defined on a cut with C, since as long as z does not lie on C, the integral is well-defined. Note that if $f(\zeta)$ is a constant, we can just evaluate this integral using the logarithm. Consider what happens if C is a closed curve. Then we have:

$$I(z) = \begin{cases} f(z), z \in C \\ 0, z \notin C \end{cases}$$

by the Cauchy integral formula provided $f(z)$ is analytic inside C. Now consider a point z_1 on the curve. Let z approach z_1 from both inside and outside the curve. Call the limiting values z_1^+ and z_1^- respectively.

When $z \to z_1^+$, we must have $I^+(z_1) = \lim_{z\to z_1^+} f(z) = f(z)$ because z still remains inside C. Similarly, as $z \to z_1^-$, we must have $I^-(z_1) = 0$ because z is now outside C.

We now define the following: $I_p(z_1) = \frac{1}{2\pi i} P\oint_C \frac{f(\zeta)d\zeta}{\zeta - z_1}$ where z_1 is on C. We hence will integrate on C and cut out a small neighbourhood around z_1. We also require $I_p(z_1) = \frac{1}{2}[I^+(z_1) + I^-(z_1)]$. The latter equation follows because we can perturb the contour slightly to either go around z_1 in the clockwise or anticlockwise fashion. But since we have from the Integration along the arc earlier, going in either direction around the pole will give a value with the same magnitude ($\phi = \pi$). Note further that we can write $f(z_1) = I^+(z_1) - I^-(z_1)$. why?

Now we can write $I^+(z_1) = I_p(z_1) + \frac{1}{2}f(z_1) = \frac{1}{2\pi i}P\oint \frac{f(\zeta)}{\zeta - z_1}d\zeta + \frac{1}{2}f(z_1)$ and $I^-(z_1) = I_p(z_1) - \frac{1}{2}f(z_1)$.

Plemelj Formula for a non-closed arc Call C_+ the arc that goes around the point z_1^+ in an anticlockwise fashion. Call C_- the arc that goes around the point z_1^- in the clockwise fashion. $I^+(z_1) = \frac{1}{2\pi i}\int_{C_+} \frac{f(\zeta)}{\zeta - z_1^+}d\zeta, I^-(z_1) = \frac{1}{2\pi i}\int_{C_-} \frac{f(\zeta)}{\zeta - z_1^-}d\zeta$. But by the Integration along the Arc formula earlier, the integration along the pole has the same magnitude in either direction. Hence we can write $I^+(z_1) = \frac{1}{2\pi i} P \int_C \frac{f(\zeta)}{\zeta - z_1} d\zeta + \frac{1}{2} f(z_1)$ and $I^-(z_1) = \frac{1}{2\pi i} P \int_C \frac{f(\zeta)}{\zeta - z_1} d\zeta - \frac{1}{2} f(z_1)$. Call $I_p(z_1) = \frac{1}{2\pi i} P \int_C \frac{f(\zeta)}{\zeta - z_1} d\zeta$ so that $I_p(z_1) = \frac{1}{2}[I^+(z_1) + I^-(z_1)]$ and $f(z_1) = I^+(z_1) - I^-(z_1)$.

Example $I(z) = \int_{-1}^{1} \frac{dt}{t-z}, z \neq x, |x| < 1$. This is a perfectly well-defined function except on the line from $z = -1$ to $z = 1$. We need to choose a branch that allows $I(z)$ to be single-valued outside the circle $|z| = 1$. We find that this branch will be $I(z) = Log(z-1) - Log(z+1)$, with angles around $z = \pm 1$ defined by $\theta \in [-\pi, \pi)$. Then, just above the cut, $z^+ = x + 0i, z^- = x - 0i$. Picking a x on the cut, we can define $I_p = P \int_{-1}^{1} \frac{dt}{t-x} = \lim_{\epsilon \to 0} \left[\int_{-1}^{x-\epsilon} \frac{dt}{t-x} + \int_{x+\epsilon}^{1} \frac{dt}{t-x} \right]$.

Chapter 9

Week 9

9.1 Monday 24 Nov 2014

Residue at infinity Define the residue at infinity to be $\text{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\int_{C_\infty} f(z)dz$, where C_∞ is the limit $R \to \infty$ of a circular contour $|z| = R$. We calculate this using the a_{-1} coefficient of the Laurent series in the neighbourhood of infinity. We make the change of variables $z = \frac{1}{t}$ so that $z = \frac{-t}{t^2}dt$ and hence we have the transformation of $C_R : z = Re^{i\theta}$ to $C_\epsilon : \frac{1}{R}e^{-i\theta} = \epsilon e^{-i\theta}$. Note that the transformed contour is now clockwise. Hence we have:

$$\text{Res}(f(z), \infty) = -\frac{1}{2\pi i}\oint_{C_\epsilon} \frac{1}{t^2} f(1/t)dt$$

which we can write (noting that C_ϵ is taken clockwise), $\text{Res}(f(z), \infty) = \text{Res}(\frac{1}{t^2}f(1/t), 0)$. Note that this is different based on how the residue at infinity is defined. In this case, we take the residue at infinity to be measured by a anticlockwise contour.

Integrals with branch points Consider the function $f(z) = \frac{z^{a-1}}{z+1}$ which has branch points at zero and infinity, as well as a simple pole at $z = -1$. We want to make this function single-valued by letting $z^{a-1} = r^{a-1}e^{i(a-1)\theta}, 0 < \theta \leq 2\pi$.

Now due to the choice of branch, the integrals on each side of the cut do not cancel. We will choose the keyhole contour that surrounds the origin and the positive real axis. Start just ϵ above the real axis, then take a anticlockwise circle around the origin with radius R, then end ϵ below the real axis, then encircle the branch point with a small circle oriented clockwise.

On the contour just above the real axis, we have $\theta = 0$, $z = r, r : \epsilon \to R$ and $dz = dr$. Similarly, on the contour just below the real axis, we have $\theta = 2\pi$, $z = re^{2\pi i}, r : R \to \epsilon$ and $dz = dre^{2\pi i}$. Hence the integral along the top of the real axis is $\int_\epsilon^R \frac{r^{a-1}}{r+1}dr$ and the integral along the bottom of the real axis is $\int_R^\epsilon \frac{(re^{2\pi i})^{a-1}}{re^{2\pi i}+1}e^{2\pi i}dr = \int_R^\epsilon \frac{r^{a-1}e^{2\pi i(a-1)}}{r+1}dr = -e^{2\pi i a}\int_\epsilon^R \frac{r^{a-1}}{r+1}dr$.

Now we want to show that the contributions of the integral on the big circle C_R and the small circle (clockwise) C_ϵ go to zero. Use the ML bound:

$$\left|\int_{|z|=R} \frac{z^{a-1}}{z+1}dz\right| \leq \frac{R^{a-1}}{R-1}2\pi R$$

note that the denominator has the "reverse" triangular inequality. This bound goes to zero as $R \to \infty$. Similarly,

$$\left|\int_{|z|=\epsilon} \frac{z^{a-1}}{z+1}dz\right| \leq \frac{\epsilon^{a-1}}{1-\epsilon}2\pi\epsilon$$

which also goes to zero as $\epsilon \to 0$. We hence have that the integral along this contour will equal $(1 - e^{2\pi i a})I$, where $I = \int_\epsilon^R \frac{r^{a-1}}{r+1}dr$. By the Cauchy residue theorem, we also have that this the integral on the contour will equal to $2\pi i$ times the residue at -1, which is $-e^{\pi i a}$. Taking $R \to \infty, \epsilon \to 0$, we have $\int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{\pi}{\sin \pi a}$.

Justification for the cut Consider two branches and contours: $f_1(z) = \frac{z^{a-1}}{z+1}, -\pi/2 < \arg z \leq 3\pi/2$ and $f_2(z) = \frac{z^{a-1}}{z+1}, \pi/2 < \arg z \leq 5\pi/2$. Consider two contours that make up the keyhole contour. The first contour C_1 will start above the real axis and end along some ray L_1 that is not just below the real axis, thereby encircling the branch point in an incomplete fashion. Choose L_1 such that C_1 encircles the point $z=-1$. The second contour "completes" the keyhole contour. By the residue theorem, we know that $\int_\epsilon^R \frac{r^{a-1}}{r+1}dr + \int_{C_{R,1}} + \int_{L_1} + \int_{C_{\epsilon,1}} = 2\pi i \text{Res}(f_1(z), -1)$ while $\int_R^\epsilon \frac{(re^{2\pi i})^{a-1}}{r+1}dr + \int_{C_{R,2}} + \int_{-L_1} + \int_{C_{\epsilon,2}} = 0$ since only the first contour encircles the point at $z=-1$. But the values of the three integrals (without the kernel stated) and the residue remains unchanged upon replacing f_1 and f_2 with another function with a different branch cut.

Example Define $I = \int_0^1 x^{a-1}(1-x)^{-a}dx, 0 < a < 1$ which has branch points at $z=0, z=1$. Then consider $f(z) = \frac{z^a}{z}(z-1)^{-a}$ and define a branch $(z-1)^a = r_1 e^{ia\theta_1}, z^a = r_2 e^{ia\theta_2}, -\pi < \theta_1, \theta_2 \leq \pi$. Then we have $z^a(z-1)^{-a} = \left(\frac{r_2}{r_1}\right)^a e^{ia(\theta_2-\theta_1)}$. Then the branch cut is between $z=0$ and $z=1$ by checking the continuity of $\theta_2 - \theta_1$ across the real axis. Hence $f(z)$ will be analytic in the plane except on the cut $x \in [0,1]$.

Now we need to define the values of r_1 and r_2 on the contour. Since the contour is restricted to the real axis from $z=0$ to $z=1$, then we have that $r_2 = x$ and $r_1 = 1-x$ along this range of the real axis. Hence on the top of the cut, we have that $z^a(z-1)^{-a} = \left(\frac{x}{1-x}\right)^a e^{-ia\pi}$ while on the bottom of the cut we have $z^a(z-1)^{-a} = \left(\frac{x}{1-x}\right)^a e^{ia\pi}$.

Now we define the contour that will integrate on. We start ϵ away from $z=0$, go around the origin in the counterclockwise fashion, head out to $z=1$ under the branch cut, go around the branch point at $z=1$ counterclockwise, then head back to the origin over the branch cut. This is the "dog bone" contour. Hence we have that:

$$\begin{aligned}\int_C f(z)dz &= \int_1^0 \frac{1}{x}\left(\frac{x}{1-x}\right)^a e^{-ia\pi}dx + \int_0^1 \frac{1}{x}\left(\frac{x}{1-x}\right)^a e^{ia\pi}dx + \int_{C,z=0} + \int_{C,z=1} \\ &= (e^{ia\pi} - e^{-ia\pi})\int_0^1 x^{a-1}(1-x)^{-a}dx \\ &= 2i\sin(\pi a)I\end{aligned}$$

where $I = \int_0^1 x^{a-1}(1-x)^{-a}dx$. Now to evaluate $\int_C f(z)dz$, we need to deform the contour C_∞ because the Cauchy theorem does not apply with the current C. We can do this because the function is analytic away from the branch cut. hence hence we have that the value of $\int_C f(z)dz$ will be $2\pi i$ times the residue at infinity (defined on a contour going counterclockwise).

9.2 26 Nov 2014 Wednesday

Plemejl formulae Recall the formulae:

$$\begin{aligned}I^+(z_1) &= \frac{1}{2\pi i}P\oint \frac{f(\zeta)d\zeta}{\zeta - z_1} + \frac{1}{2}f(z_1) \\ I^-(z_1) &= \frac{1}{2\pi i}P\oint \frac{f(\zeta)d\zeta}{\zeta - z_1} - \frac{1}{2}f(z_1)\end{aligned}$$

Hilbert Transform This transform can be obtained from the Plemejl formulae. Suppose we have $f(z) = u + iv$ which is analytic on the upper half-plane ($y \geq 0$) and $f(z) \to 0$ as $z \to \infty$ in the upper half plane. Now consider the semicircular contour containing a certain point z. Then by the Cauchy integral formulae. we have that:

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(\zeta)d\zeta}{\zeta - z}$$

along the real axis when we let the semicircle go to infinity. Now on $y=0$, we have that $f(\zeta) = u(\zeta,0) + iv(\zeta,0) = U(\zeta) + iV(\zeta)$. Now let $z \to x + 0i$, just above the real axis. Then using the Plemejl formula, we have that:

$$U(x) + iV(x) = \frac{1}{2\pi i}P\int_{-\infty}^\infty [U(\zeta) + iV(\zeta)]\frac{1}{\zeta - x}d\zeta - \frac{1}{2}[U(x) + iV(x)]$$

Comparing real and imaginary parts, we have that:

$$\begin{aligned}U(x) &= \frac{1}{\pi}P\int_{-\infty}^\infty \frac{V(\zeta)}{\zeta - x}d\zeta \\ V(x) &= \frac{-1}{\pi}P\int_{-\infty}^\infty \frac{U(\zeta)}{\zeta - x}d\zeta\end{aligned}$$

these two integrals above are called a **pair of Hilbert transforms**. Hence, given a function $U(x)$, we can form a function $V(x)$ whose inverse will be $U(x)$.

Integral along a branch point involving principal values Consider $I(a) = P\int_0^\infty \frac{x^{a-1}}{1-x}dx, 0 < a < 1$. Now since $a \notin \mathbb{Z}$, we expect branch points at $z = 0, \infty$ and a pole at $z = 1$. Pick a branch cut along the positive real axis that cuts through the singularity.

We hence define the following contour: Start near the origin above the branch point, go to $z = 1$ above the branch point, go in a clockwise semicircle around the singularity at $z = 1$, continue to $z = R$, go around a large circle of radius R until you reach the bottom of the branch cut, head back to $z = 1$ under the branch cut, do a clockwise semicircle around the singularity, the proceed to the origin under the branch cut, then go around the branch point at the origin in the clockwise fashion with radius ϵ back to the starting point.

Now the closed contour does not enclose any singularities, hence we know that $\oint_C f(z)dz = 0$. We know that along the circular parts, the function will go to zero under the limits $R \to \infty$ and $\epsilon \to 0$. Then we have:

$$0 = \left(\int_\delta^{1-\epsilon} \frac{x^{a-1}}{1-x}dx + \int_{1+\epsilon}^R \frac{x^{a-1}}{1-x}dx\right)(1-e^{2\pi i(a-1)}) + \int_{\text{bottom of semicircle}} \frac{z^{a-1}}{1-z}dz + \int_{\text{top of semicircle}} \frac{z^{a-1}}{1-z}dz$$

where we note that $\frac{(xe^{2\pi i})^{a-1}}{1-(xe^{2\pi i})}dx = e^{2\pi i(a-1)}\frac{x^{a-1}}{1-x}dx$.

Now on the top of the semicircle, we write $z = 1+\epsilon e^{i\theta}$ so that $z^{a-1} = \left|1+\epsilon e^{i\theta}\right|^{a-1} e^{i(a-1)\arg z}$. Since $\arg z = 0$ as $\epsilon \to 0$, $z^{a-1} = 1 + O(\epsilon)$, and hence $z^{a-1} \to 1$ as $\epsilon \to 0$.

On the bottom of the semicircle, however, the argument of z will be 2π, and hence we have that $z^{a-1} = \left|1+\epsilon e^{i\theta}\right|^{a-1} e^{i(a-1)2\pi i} = e^{2\pi i(a-1)} + O(\epsilon)$.

Now the residue on the top of the cut will be -1 and the residue at the bottom of the cut will be $-e^{2\pi i(a-1)}$. Hence the integral on the two half-poles will equal:

$$\int_{\text{top of semicircle}} \frac{x^{a-1}}{1-x}dx + \int_{\text{bottom of semicircle}} \frac{x^{a-1}}{1-x}dx = -i\pi(-1) - i\pi(-e^{2\pi i(a-1)})$$

where the $-i\pi$ comes from the fact that each integral is a half-pole and goes in the clockwise direction.

hence in the limit as $R \to \infty$, $\epsilon \to 0$, $\delta \to 0$, we have that:

$$0 = P\int_0^\infty \frac{x^{a-1}}{1-x}dx\left(1-e^{2\pi i(a-1)}\right) + i\pi\left(1+e^{2\pi i(a-1)}\right)$$

and hence:

$$I(a) = P\int_0^\infty \frac{x^{a-1}}{1-x}dx = \pi\cot(a\pi)$$

Infinite integrals/Improper integrals Consider the integral:

$$f(z)\int_{C_\infty} F(z,\zeta)d\zeta$$

which is a function of an integral evaluated on the infinite circle. Consider the limiting case when:

$$f_R(z) = \int_{C_R} F(z,\zeta)d\zeta$$

If we can show that $\lim_{R\to\infty} f_R(z) = f(z)$, then we say that the integral is convergent. A stronger condition is uniform convergence. If there exists an $R_0(\epsilon)$ such that $|f(z) - f_R(z)| < \epsilon$ for $R > R_0$ and for all $z \in D$.

If the integral is uniformly convergent, then the following are true:

- $\lim_{R\to\infty} f_R'(z) = f'(z)$
- $f_R'(z) = \int_{C_R} \frac{\partial F}{\partial z} d\zeta$ when we exchange the operation of differentiation and integration.
- $f(z)$ is analytic.

Explicitly, when the infinite integral is uniformly convergent:

$$\lim_{R\to\infty} \int_{C_R} \frac{\partial F}{\partial z} d\zeta = \int_{C_\infty} \frac{\partial F}{\partial z} d\zeta = \frac{d}{dz} \int_{C_\infty} F(z,\zeta) d\zeta = f'(z)$$

Example: Gamma function Define (using t instead of ζ):

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

integrating by parts, we obtain the recursion formula:

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

It is also true that $\Gamma(n) = (n-1)!, n \in \mathbb{N}$ because $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$ and we can use the recursion relationship to obtain other integer values.

Note that

$$|t^{z-1}e^{-t}| \leq t^{x-1}e^{-t}, x = \Re(z)$$

Hence in the limit where $R \to \infty$:

$$\lim_{R\to\infty} \int_0^R t^{x-1} e^{-t} dt = \int_0^\infty t^{x-1} e^{-t} dt$$

is uniformly convergent because of the e^{-t} part that will dominate all t^{x-1} for all $x > 0$. Then by the comparison test with the above integral, the gamma function:

$$\lim_{R\to\infty} \int_0^R e^{-t} t^{z-1} dt$$

converges uniformly and is equal to the gamma function for $\Re(z) > 0$. This implies that $\Gamma(z)$ is an analytic function of z for $\Re(z) > 0$.

Beta function The complete beta function is defined:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$$

With $\Re(p), \Re(q) > 0$.
It can be shown that $\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Chapter 10

Week 10

10.1 Monday 1 Dec 2014

Gamma Function Recall that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ which is absolutely convergent for $\Re(z) > 0$. Also, $\Gamma(z+1) = z\Gamma(z)$. We can write this as $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ which will be singular at $z = 0$.

Negative z values for Gamma Function Note that $\int_0^\infty e^{-3/2} e^{-t} dt$ does not converge. This should correspond to $\Gamma(\frac{1}{2})$. Consider the strip $-1 < \Re(z) \leq 0$. We know that $0 < \Re(z+1) \leq 1$ and $\Gamma(z+1)$ is defined by a convergent integral. Hence we can just use the recurrence relation to define $\Gamma(-1/2) = \Gamma(1/2)/(-1/2)$. Note that this does not work for $z = 0$ because the recurrence relation requires a division by z. We can repeat this process for strips further to the left in the real axis. This is the analytic continuation of the gamma function into the left half plane. Hence we can define Γ to be analytic in the whole z plane except at he poles of $z = -n, n \in \mathbb{Z}_0^+$.

Conformal Mapping Motivation: Solving Laplace's equation in some domain $\nabla^2 \phi = 0$ with $(x, y) \in D$, where ϕ is given on ∂D or $\frac{\partial \phi}{\partial n}$ is given on ∂D (or both boundary conditions).

Set-up Let $w = f(z)$ be a mapping from D in the xy plane to $\mathcal{D}$ in the uv plane. Write $W(x, y) = U(x, y) + iV(x, y)$.

Exponential Mapping Consider $w = e^z$ and its inverse $z = \text{Log}\, w$. This function maps the strip in the xy plane: $-\pi \leq y \leq \pi$ to the entire w plane with a branch cut along the negative real axis in the w plane. The mapping is one to one.

Now note that the mapping $z = \text{Log}\, w$ can be written as $z = x + iy = \ln(u^2 + v^2)^{1/2} + i \tan^{-1} \frac{v}{u} \implies x = \ln(u^2 + v^2)^{1/2}$, $y = \tan^{-1} \frac{v}{u}$. Hence the lines $x = c$, c constant maps into $u^2 + v^2 = e^x$, which is a circle of radius $e^{x/2}$. Similarly, the lines $y = c$, c constant maps into radial lines through the origin ($v = u \tan y$).

For the inverse mapping, we write $w = e^z = e^x \cos y + ie^x \sin y$ so $u = e^x \cos y, v = e^x \sin y$. Hence the line $u = c$, c constant, maps to the function $x = -\ln(\cos y/u)$ and the line $v = c$, c constant, maps to the curve $x = -\ln(\sin y/v)$.

Theorems

- **Locally One to Oneness** Suppose $w = f(z)$ is analytic in some domain near $z = z_0$. Let $f'(z_0) \neq 0, \infty$. Then there exists a neighbourhood N around z_0, such that the mapping is locally one-to-one in the neighbourhood.
  - **Sketch of Proof** Since w is analytic at z_0 its derivative is also analytic there. Then we can write the Taylor series $w = f(z_0) + f'(z_0)(z - z_0) + \ldots \implies w - w_0 = f'(z_0)(z - z_0)$. Hence very close to z_0, we have the mapping between $w - w_0$ and $z - z_0$ that is one-to-one. It is not singular because $f'(z_0) \neq 0$.
  - Note that a function can be locally one-to-one everywhere but may not be globally one-to-one. For example, the mapping $w = z^2, z = w^{1/2}$ is locally one-to-one everywhere except at $z = w = 0$ and along a branch cut.
  - A linear mapping is globally one-to-one. $f(z) = 1/z, f(z) = Az + B$ are globally one-to-one.
- **Conformality** Consider a point z_0 in the xy plane and $w_0 = f(z_0)$ in the uv plane. Consider two curves passing through z_0, and call these curves γ_1, γ_2. Let the curves make an angle θ to each other at z_0. Now consider the image of these curves in the uv plane. Let the angle between the images of the curves in the uv plane be θ'. A mapping at $z = z_0$ is conformal if $\theta = \theta'$ for all curves passing through z_0. An analytic function is **conformal** at every point where its derivative is nonzero and noninfinite.

– **Sketch of proof** Write $w = f(z)$ so $dw = f'(z_0)dz$ in a small epsilon neighbourhood around $f(z_0)$. Let $dw_1 = f'(z_0)dz_1, dw_2 = f'(z_0)dz_2$ for two small displacements dz_1, dz_2 in the xy plane. Then we can write $\arg(dw_1) = \arg f'(z_0) + \arg dz_1$ and $\arg(dw_2) = \arg f'(z_0) + \arg dz_2$. Hence we have that $\arg dw_2 - \arg dw_1 = \arg dz_2 - \arg dz_1 \implies \theta = \theta'$.

Some Elementary Mappings

- $w = Az$, $A = ae^{i\alpha}, a, \alpha \in \mathbb{R}$. Then this is a stretching by factor a and rotation by angle α.
- $w = Az + b$. Same as above but with a translation b.

10.2 Wednesday 03 Dec 2014

Inversion transformation Consider $w = \frac{1}{z}$. Maps the exterior of the unit circle to the interior and vice versa. Mapping is globally one-to-one but does not preserve shapes. Examples of transformations under inversion:

- A straight line through the origin in the xy plane maps to a straight line passing through the origin in the uv plane.
- A straight line not through the origin in the xy plane can be written as $Ax + By = C, C \neq 0$ and maps to $u^2 + v^2 - \frac{A}{c}u + \frac{B}{c}v = 0$ which is a circle passing through the origin. The point at infinity maps to the origin. The point on the line closest to the origin maps to the point on the circle furthest away from the origin.
- Circles through the origin in the z-plane map to straight lines not passing through the origin in the uv plane (just the reverse of the previous statement).
- Circles not passing through the origin in the xy plane map to circles not passing through the origin in the uv plane. Note that if we start with two concentric circles in the xy plane that do not pass through the origin, then under the transformation the images are no longer concentric. The interior of the inner circle and the exterior of the outer circle in the xy plane are mapped to the interiors of the image circles in the uv plane.

Generally, the class of straight lines and circles is mapped to the class of straight lines and circles under the inversion mapping and the mapping is one-to-one.

Mobius transformation Consider $w = f(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad \neq bc$. Restriction is required to that w is not just a constant. $f'(z) = \frac{ad-bc}{cz+d)^2}$ hence the mapping is conformal at every point except at the pole at $z = \frac{-c}{d}$. This Mobius transformation can be written as the composition of the following mappings:

$$W_1 = cz + d$$
$$W_2 = \frac{1}{W_1}$$
$$W_3 = (b - \frac{ad}{c})W_2 + \frac{a}{c}$$
$$W = W_3 \circ W_2 \circ W_1 = \frac{az+b}{cz+d}$$

Notes on the Mobius transformation

- This transformation maps the class of circles and lines back to itself.
- This transformation is 1-1.
- This transformation is conformal everywhere except at the pole $z = \frac{-c}{d}$.
- The inverse transformation is $z = f^{-1}(w) = \frac{-dw+b}{cw-a}$ which is also a Mobius transformation.
- A composition of Mobius transformations is also a Mobius transformation.
- The mapping of three non-coincident pairs of points define a unique Mobius transformation.

Application of Mobius Transformations Consider two cylinders C_1 and C_2, one of radius R_1 centred at the origin and the other cylinder of radius $R_2 < R_1$ inside the first cylinder passing through the origin. We want to solve the heat equation $\frac{\partial T}{\partial t} = \kappa \nabla^2 T, T = T(x, y, t)$. We keep the big cylinder at temperature T_1 and the inner cylinder at temperature T_2. Now we want to find the temperature in between the cylinders. Consider the steady-state so that the time derivative vanishes. Hence we just need to solve $\nabla^2 T = 0$ with the Dirichlet boundary conditions at the cylinders. We first introduce non-dimensional variables $\psi = \frac{T-T_2}{T_1-T_2}$ such that $\psi = 1$ at C_1 and $\psi = 0$ at C_2. We also introduce the variables $x = \frac{X}{R_1}, y = \frac{Y}{R_1}$. Now also consider for simplicity $a = \frac{R_2}{R_1} = \frac{1}{3}$. Now define the function $w(z) = \phi + i\phi$. If it is analytic, then the real and imaginary parts are harmonic. We hence want to find the function w to satisfy the boundary conditions. We pick the following mapping: $\zeta = \frac{z-1/3}{z-3}$. Note that on the xy-plane, the centre of the inner circle is at $z = a$. The mapping maps the two circles into the $\zeta = \xi + i\eta$ plane with the images of the two circles being concentric. The outer circle is mapped to the circle in the $\xi\eta$ plane $|\zeta| = \frac{1}{3}$ and the inner circle is mapped to the circle $|\zeta| = \frac{1}{9}$. Now we can solve the problem in the ζ plane. Consider the function $G(\zeta) = \Phi + i\Psi$. We know that $\nabla^2\Phi = \nabla^2\Psi = 0$ and Laplace's equation is invariant under the mapping. Hence we want to find Ψ such that $\Psi = 0$ in the inner circle $|\zeta| = \frac{1}{9}$ and $\Psi = 1$ in the outer circle $|\zeta| = \frac{1}{3}$. By inspection, we claim that the solution is $G(\zeta) = \frac{i\log(9\zeta)}{\ln 3}$ which is an analytic function of ζ in the annulus. Taking the imaginary part of G, we obtain that $\Psi = \frac{\ln|9\zeta|}{\ln 3}$. On the outer circle, we calculate that $\Psi(|\zeta| = \frac{1}{3}) = \frac{\ln(9\times\frac{1}{3})}{\ln 3} = 1$ and $\Psi(|\zeta| = \frac{1}{9}) = 0$ so the boundary conditions are satisfied. To get the solution in the xy plane, we just substitute $\psi(z) = \Im[w(z)] = \Im\{G[\zeta(z)]\}$.

10.3 Friday 5 Dec 2014

2D Irrotational, invoiced flow past body Consider $\rho = c$ constant and $\mu = 0$. Consider incoming flow with horizontal component U_∞ and vertical component V_∞ so that it has angle of incident α. Since the flow is 2D, then there exists a stream function (volume flow) $\psi(x, y, t)$ such that the fluid velocity in the x-direction $u = \frac{\partial\psi}{\partial y}$ and the fluid velocity in the y-direction $v = \frac{-\partial\psi}{\partial x}$. Irrotational means that the fluid vorticity $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is zero. Combing these two conditions, we obtain that $\nabla^2\psi = 0$. This also implies the existence of some ϕ such that $\nabla^2\psi = 0$ and $u = \frac{\partial\phi}{\partial x}, v = \frac{\partial\phi}{\partial y}$. Mass continuity also require that $\nabla \cdot (u, v) = 0$. We may hence write:

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$
$$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

but these are just the CR equations! Hence ϕ and ψ are harmonic conjugates and there exists a complex velocity potential $W(z)$ such that $W = \phi + i\psi$. The complex velocity can be expressed as a derivative $u - iv = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}$ and equivalent representations of $\frac{dW}{dz}$.

The problem is hence this: Given a body shape, find $W(z)$ satisfying (a) $\frac{dW}{dz} \to U_\infty + iV_\infty$ as $z \to \infty$ for all $\arg z$ and (b) $\frac{v}{u}|_{body} = \frac{dy}{dx}|_{body}$ and (c) ψ =constant along the body. We also have Bernoulli's principle: $H = \frac{P}{\rho} + \frac{1}{2}(u^2 + v^2)$ =constant= $\frac{P_\infty}{\rho} + \frac{1}{2}(U_\infty^2 + V_\infty^2)$.

Example: Flat Plate Aerofoil Let the aerofoil extend from $x = -2a$ to $x = 2a$ and let it lies on the real axis. We pick the conformal mapping $z = \zeta + \frac{a^2}{\zeta}$ or equivalently $\zeta = \frac{z+(z^2-4a^2)^{1/2}}{2}$ (conformal everywhere except at $z = \pm 2a$). The inverse is multivalued and hence we need to pick a branch. It has branch points at $z = \pm 2a$. We pick the branch that maps the exterior of the aerofoil to the exterior of the unit circle $|\zeta| = 1$. We write $z = re^{i\theta}$ and $\zeta = \rho e^{i\chi}$. Then, substituting these back into the mapping, we have:

$$re^{i\theta} = \rho e^{i\chi} + \frac{a^2}{\rho}e^{-i\chi}$$

Choosing $\rho = a$, we obtain that:

$$x + iy = a(e^{i\chi} - e^{-i\chi}) = 2a\cos\chi$$

Hence $x = 2a\cos\chi$ and $y = 0$.

Consider $w(z) = U_\infty z, \frac{dw}{dz} = U_\infty$ which is just uniform flow. We hence write $w = U\infty(x + iy)$ and hence $\phi = U_\infty x$ and $\psi = U_\infty y$.

Now consider $w = w(z(\zeta))$ with the mapping previously. This corresponds to flow around a cylinder in the mapped plane. We hence write $w(z) = U_\infty(\zeta + \frac{a^2}{\zeta})$. Now we consider a rotation of axes in the ζ plane to make the flow come at an angle to the horizontal ξ axis. We hence make the additional mapping $\zeta_1 = \zeta e^{i\alpha}$ which is just a rotation in α. We also add a circulation Γ around the origin in the $\xi - \eta$ plane. Then we will be moving the two circulation points (where $\frac{dW}{d\zeta} = 0$) at the surface of the cylinder. Taking all these changes into account, we hence have:

$$W(\zeta) = U_\infty(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}}) - \frac{\Gamma}{2\pi i}\text{Log}\frac{\zeta}{a}$$

where the first term indicates the rotation $e^{i\alpha}$ and the second term is the circulation added. We note that on $|\zeta| = a$, the imaginary part of $w(\zeta)$ will be a constant (actually zero). Returning to the z plane, we write:

$$w(z) = w(\zeta(z)) = U_\infty \left[e^{-i\alpha}\left(\frac{z + (z^2 - 4a^2)^{1/2}}{2}\right) + a^2 e^{i\alpha}\frac{2}{z + (z^2 - 4a^2)^{1/2}}\right] - \frac{\Gamma}{2\pi i}\text{Log}\frac{z + (z^2 - 4a^2)^{1/2}}{2}$$

It can be shown that on any contour around the aerofoil, we will have $\oint \vec{u} \cdot d\vec{l} = \Gamma$. We want to choose Γ such that the flow off the lagging edge comes off smoothly. This is called the Kutta condition. When $z \to \pm 2a$, then $\frac{dw}{dz} = (z \pm 2a)^{-1/2} + \ldots$. Then it is singular at $z \pm 2a$. We want to relieve the singularity at one point (not possible for both).

Using the chain rule, we write:

$$\frac{dW}{dz} = \frac{dW}{d\zeta}\frac{d\zeta}{dz}$$

where

$$\frac{dz}{d\zeta} = 1 - \frac{a^2}{\zeta^2} = \frac{\zeta^2 - a^2}{\zeta^2}$$

hence $\frac{dW}{dz} = \frac{dW}{d\zeta}\frac{\zeta^2}{\zeta^2 - a^2}$. To ensure that the velocity does not blow up at $\zeta \pm a$, we want to find Γ such that $\frac{dW}{d\zeta} = 0$ so that $\frac{dW}{dz}$ at $\zeta = a$ is finite. We hence evaluate $\frac{dW}{d\zeta}$:

$$\frac{dW}{d\zeta} = U_\infty\left(e^{-i\alpha} - \frac{a^2 e^{i\alpha}}{\zeta^2}\right) - \frac{\Gamma}{2\pi}\frac{1}{\zeta}$$

hence when $\zeta = a$,

$$U_\infty(-2i\sin\alpha) - \frac{\Gamma}{2\pi i a} = 0 \implies \Gamma = 4\pi a U_\infty \sin\alpha$$

The choice of Γ above satisfies the Kutta condition. There is still a singularity at the leading edge, although we have relieved the singularity at the lagging edge. The singularity at the leading edge can be relieved by rounding the leading edge.

It can also be shown that the drag is zero and the lift is proportional to $\sin\alpha$ where the drag is in the direction of U_∞ and the lift is orthogonal to it.

Schwarz-Christoffel Mappings This is a mapping that maps the interior of polygon shapes in the z-plane into the upper half of the zeta plane.