Electrodynamics

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1 Fundamental Mathematics

1.1 Vector Derivatives

- Cartesian

$$\nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \quad (1)$$

- Spherical

$$\vec{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \quad (2)$$

$$d\tau = r^2 \sin\theta dr d\theta d\phi \quad (3)$$

$$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi} \quad (4)$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \quad (5)$$

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} r v_\phi\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi} \quad (6)$$

$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 t}{\partial \phi^2} \quad (7)$$

- Cylindrical

$$\vec{dl} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z} \quad (8)$$

$$\nabla t = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z} \quad (9)$$

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \quad (10)$$

$$\nabla \times \vec{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} \quad \frac{1}{s}\left(\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right)\hat{z} \quad (11)$$

$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \quad (12)$$

1.2 Vector Identities

- Triple Products

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad (13)$$

$$[\vec{A} \times (\vec{B} \times \vec{C})] + [\vec{B} \times (\vec{C} \times \vec{A})] + [\vec{C} \times (\vec{A} \times \vec{B})] = 0 \quad (14)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \quad (15)$$

- Gradients

$$\nabla(fg) = f\nabla g + g\nabla f \quad (16)$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} \quad (17)$$

- Divergences

$$\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f \quad (18)$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \quad (19)$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \quad (20)$$

- Curls

$$\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \quad (21)$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) \quad (22)$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A \quad (23)$$

$$\nabla \times (\nabla f) = 0 \quad (24)$$

1.3 Fundamental Theorems

- Fundamental Theorem of Calculus

$$\int_a^b (\nabla f) \cdot \vec{dl} = f(b) - f(a) \quad (25)$$

- Stokes' Theorem

$$\int (\nabla \times \vec{A}) \cdot \vec{da} = \oint_C \vec{A} \cdot \vec{dl} \quad (26)$$

- Divergence Theorem

$$\int (\nabla \cdot \vec{A}) d\tau = \oint_S \vec{A} \cdot \vec{da} \quad (27)$$

1.4 Additional Vector Formulae

- Vector Area of S

$$\vec{a} \equiv \int_S \vec{da} \tag{28}$$

$$\vec{a} = \frac{1}{2}\oint \vec{r} \times \vec{dl} \tag{29}$$

- Rotation Matrix

$$\begin{pmatrix} A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix} \tag{30}$$

- Infinitesimal Displacement

$$dT = \nabla T \cdot \vec{dl} \tag{31}$$

- Laplacian

$$\nabla^2 \vec{v} \equiv (\nabla \cdot \nabla)\vec{v} \tag{32}$$

- For any vector c, where a is the area vector of the surface

$$\oint_S (\vec{c} \cdot \vec{r})\vec{dl} = \vec{a} \times \vec{c} \tag{33}$$

1.5 Dirac Delta Function

- Uniqueness

$$D_1(x) = D_2(x) \iff$$
$$\int_{-\infty}^{\infty} f(x)D_1(x)dx = \int_{-\infty}^{\infty} f(x)D_2(x)dx \tag{34}$$

- Even

$$\delta(-x) = \delta(x) \tag{35}$$

- Under integrals

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{36}$$

$$\iiint_{\text{all space}} f(\vec{r})\delta^3(\vec{r} - \vec{a})d\tau = f(\vec{a}) \tag{37}$$

- Under differentials

$$x\frac{d}{dx}\delta(x) = -\delta(x) \tag{38}$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r}) \tag{39}$$

$$\nabla^2 \frac{1}{\eta} = -4\pi\delta^3(\eta) \tag{40}$$

1.6 Potentials

- Zero curl implies gradient

$$\nabla \times \vec{F} = 0 \iff \vec{F} = -\nabla V \tag{41}$$

- Zero divergence implies curl

$$\nabla \cdot \vec{F} = 0 \iff \vec{F} = \nabla \times \vec{A} \tag{42}$$

1.7 Time Averages

- For f, g having same wavenumber and frequency

$$< fg >= \frac{1}{2}Re(\tilde{f}\tilde{g}^*) \tag{43}$$

1.8 Trigonometry

$$\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q) \tag{44}$$

$$\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q) \tag{45}$$

$$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q) \tag{46}$$

2 Electrostatics

2.1 Electric Fields

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\eta^2}\hat{\eta}, \vec{\eta} = \vec{r} - \vec{r'} \tag{47}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\eta_i^2}\hat{\eta_i^2} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\eta}}{\eta^2}dq \tag{48}$$

2.2 Relation to potential

$$\vec{E} = -\nabla V \tag{49}$$

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot \vec{dl} \tag{50}$$

$$V = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{\eta} \tag{51}$$

2.3 Under differential operators

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = -\nabla^2 V \tag{52}$$

$$\nabla \times \vec{E} = 0 \tag{53}$$

2.4 Earnshaw's Theorem

Note that since $\nabla^2 V = 0$ in free space, $\nabla \cdot \vec{F} = 0$, implying no stationary points. Hence no stable equilibrium configuration.

2.5 Electric Flux

$$\Phi_E = \int_S \vec{E} \cdot \vec{da} \tag{54}$$

$$\oint_S \vec{E} \cdot \vec{da} = \frac{Q_{enc}}{\epsilon_0} \tag{55}$$

2.6 Boundary Conditions

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n} \tag{56}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{57}$$

$$E^{\parallel}_{above} = E^{\parallel}_{below} \tag{58}$$

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n} \tag{59}$$

2.7 Work

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i) \tag{60}$$
$$= \frac{1}{2}\int \rho V d\tau \tag{61}$$
$$= \frac{1}{2}\int V dq \tag{62}$$

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau \tag{63}$$

2.8 Green's Reciprocity Theorem

Given ρ_1, which produces V_1, and ρ_2, which produces V_2 in two different situations,

$$\int_{\text{all space}} \rho_1 V_2 d\tau = \int_{\text{all space}} \rho_2 V_1 d\tau \tag{64}$$

Proof: $\int \vec{E_1}\cdot\vec{E_2}d\tau$ is evaluated using $\vec{E_{1,2}} = -\nabla V_{1,2}$. Integrate by parts in two different ways, then equate. Surface integral vanishes when integrated to infinity.

2.9 Conductors

On a patch of charged conducting surface, where $\vec{f}$ = force per unit area,

$$\vec{f} = \sigma\left(\frac{E_{above} + E_{below}}{2}\right) \tag{65}$$

2.10 Capacitance

$$Q = CV \tag{66}$$
$$W = \frac{1}{2}CV^2 \tag{67}$$

3 Special Techniques

3.1 Fourier Trick

Multiply by an orthogonal function and sum/integrate across the space. All other terms are eliminated except for the $n = n'$ case.

$$\int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi\ y}{a}dy = \begin{cases} 0, & n' \neq n \\ \frac{a}{2}, & n' = n \end{cases} \tag{68}$$

Note also that

$$\cos(n\pi) = (-1)^n \tag{69}$$

Condition for orthogonality:

$$\int_0^a f_n(y) f_{n'}(y) dy = 0 \quad \text{for } n' \neq n \tag{70}$$

3.2 Spherical Equations for Laplacians

Radial differential equation:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = l(l+1) \tag{71}$$

is solved by

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \tag{72}$$

and Angular differential equation:

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = -l(l+1) \tag{73}$$

is solved by

$$\Theta(\theta) = P_l(\cos\theta) \tag{74}$$

where $P_l(x)$ is the Legendre Polynomial.

3.3 Legendre Polynomials

- Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \tag{75}$$

- Special Values

$$P_l(1) = 1 \tag{76}$$
$$P_l(-1) = (-1)^l \tag{77}$$

- Values

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \\ P_4(x) &= (35x^4-30x^2+3)/8 \\ P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned} \tag{78}$$

- Orthogonality

$$\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \begin{cases} 0, & l' \neq l \\ \frac{2}{2l+1}, & l' = l \end{cases} \tag{79}$$

3.4 Laplace Equation solution in spherical coordinates

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & r \leq R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & r \geq R \end{cases} \tag{80}$$

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{81}$$

Alternatively,

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}}\int_0^\pi \sigma_0(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{82}$$

$$B_l = A_l R^{2l+1} \tag{83}$$

As for surface charges,

$$\frac{\sigma_0(\theta)}{\epsilon_0} = \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) \tag{84}$$

3.5 Multipole Expansion

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(cos\theta')\rho(r')d\tau' \tag{85}$$

Considering the first few terms,

$$\begin{aligned} V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Bigg[&\frac{1}{r}\int \rho(r')d\tau' \quad \text{monopole} \\ &+ \frac{1}{r^2}\int r'\cos\theta'\rho(r')d\tau' \quad \text{dipole} \\ &+ \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(r')d\tau' + \ldots\Bigg] \quad \text{quadrupole} \end{aligned} \tag{86}$$

3.5.1 Electric Dipole

- Electric Dipole Moment

$$\vec{p} = \int \vec{r'}\rho(r')d\tau' \tag{87}$$

When displaced by vector $\vec{a}$

$$\bar{\vec{p}} = \vec{p} - Q\vec{a} \tag{88}$$

- Potential distribution

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \tag{89}$$

- Electric Field Distribution For dipole $\vec{p}$ located at origin and pointing in $\hat{z}$,

$$\vec{E}_{\text{dipole}} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{90}$$

Generally,

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \tag{91}$$

- Average field in sphere due to internal charge

$$\vec{E}_{vec} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{R^3} \tag{92}$$

$$\vec{p} = \text{Total dipole moment}$$

4 Electric Fields in Matter

4.1 Dielectrics

Induced dipole moment is approximately proportional to the field:

$$\vec{p} = \alpha\vec{E}, \alpha = \text{Atomic Polarizability} \tag{93}$$

$$\vec{\tau} = \vec{p}\times\vec{E} \tag{94}$$

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} \tag{95}$$

$$U = -\vec{p}\cdot\vec{E} \tag{96}$$

For two dipoles,

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\vec{p_1}\cdot\vec{p_2} - 3(\vec{p_1}\cdot\hat{r})(\vec{p_2}\cdot\hat{r})\right] \tag{97}$$

For a single dipole,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\eta}\cdot\vec{p}}{\eta^2} \tag{98}$$

4.2 Bound Charges

Let $\vec{P}$ be the dipole moment per unit volume.

$$\sigma_b = \vec{P}\cdot\hat{n} \tag{99}$$

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{\chi_E}{1+\chi_E}\rho_f \tag{100}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{\eta}da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{\eta}d\tau' \tag{101}$$

4.3 Macroscopic Field

The average field across any shape due to internal charge is equal to the field at the center of a uniformly polarized shape with the same total dipole moment.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\eta}\cdot\vec{P}(r')}{\eta^2}d\tau' \tag{102}$$

4.4 Electric Displacement

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P} \tag{103}$$

$$\nabla\cdot\vec{D} = \rho_f \tag{104}$$

$$\oint \vec{D}\cdot d\vec{a} = Q_f \tag{105}$$

4.4.1 Boundary Conditions

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \tag{106}$$

$$\vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}} \tag{107}$$

which is a consequence of $\vec{E}^{\parallel}_{\text{above}} = \vec{E}^{\parallel}_{\text{below}}$.

4.4.2 Linear Dielectrics

Polarization is proportional to the field:

$$\vec{P} = \epsilon_0\chi_0\vec{E} \tag{108}$$

$$\epsilon \equiv \epsilon_0(1+\chi_E) \tag{109}$$

such that

$$\vec{D} = \epsilon\vec{E} \tag{110}$$

Also, define relative permittivity ϵ_r as

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_E \tag{111}$$

When involving capacitors,

$$C_{\text{new}} = \epsilon_r C_{\text{vacuum}} \tag{112}$$

4.4.3 Boundary Conditions involving Linear Dielectrics

$$\epsilon_{\text{above}}E^{\perp}_{\text{above}} - \epsilon_{\text{below}}E^{\perp}_{\text{below}} = \sigma_f \tag{113}$$

or,

$$\epsilon_{\text{above}}\frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}}\frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f \tag{114}$$

$$V_{\text{above}} = V_{\text{below}} \tag{115}$$

4.4.4 Clausius-Mossotti Formula

For uniform,, non-polar, polarizable atoms. Atomic polarizability α is related to N, number density and ϵ_r by

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \tag{116}$$

4.4.5 Langeuin Formula

For polar substances, where P is the polarization, p is the permanent dipole moment of a single molecule, E is the external electric field, N is the number density of polar molecules.

$$P = Np\left(\coth\frac{pE}{kT} - \frac{kT}{pE}\right) \tag{117}$$

4.4.6 Work in linear dielectrics

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau \tag{118}$$

5 Magnetostatics

5.1 Fundamentals

$$\vec{F} = Q(\vec{v}\times\vec{B}) \tag{119}$$

$$P = QBR \tag{120}$$

which holds for relativistic situations too.

5.2 Currents and Forces

$$\vec{I} = \lambda\vec{v} \tag{121}$$

$$\vec{K} = \frac{d\vec{I}}{dl_\perp} = \sigma\vec{v} \tag{122}$$

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \rho\vec{v} \tag{123}$$

$$\vec{F} = I\int d\vec{l}\times\vec{B} \tag{124}$$

$$\vec{F} = \int(\vec{K}\times\vec{B})da \tag{125}$$

$$\vec{F} = \int(\vec{J}\times\vec{B})d\tau \tag{126}$$

Conservation of charge:

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \tag{127}$$

5.3 Generation of Magnetic Field

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l'}\times\hat{\eta}}{\eta^2} &(128)\\ &= \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r'})\times\hat{\eta}}{\eta^2}da' &(129)\\ &= \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r'})\times\hat{\eta}}{\eta^2}d\tau' &(130)\end{aligned}$$

5.4 Properties of Magnetic Field

Ampere's Law:

$$\nabla\times\vec{B} = \mu_0\vec{J} \iff \oint\vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \tag{131}$$

$$\nabla\cdot\vec{B} = 0 \tag{132}$$

implying the magnetic field $\vec{B}$ can be expressed as a curl of another vector field $\vec{A}$.

5.4.1 Boundary Conditions

$$B^\perp_{\text{above}} = B^\perp_{\text{below}} \tag{133}$$

$$B^\parallel_{\text{above}} - B^\parallel_{\text{below}} = \mu_0 K \tag{134}$$

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K}\times\hat{n}) \tag{135}$$

5.5 Magnetic Vector Potential

$$\vec{B} \equiv \nabla\times\vec{A} \tag{136}$$

Under Coulomb Gauge,

$$\nabla\cdot\vec{A} = 0 \tag{137}$$

$$\vec{A} = \frac{1}{4\pi}\int\frac{\vec{B}\times\hat{\eta}}{\eta^2}d\tau' \tag{138}$$

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\int\frac{\vec{I}dl'}{\eta} &(139)\\ &= \frac{\mu_0}{4\pi}\int\frac{\vec{K}}{\eta}da' &(140)\end{aligned}$$

Direction of $\vec{A}$ is usually the direction of the current.

5.5.1 Boundary Conditions

$$\vec{A}_{\text{above}} = \vec{A}_{\text{below}} \tag{141}$$

$$\oint\vec{A}\cdot d\vec{l} = \Phi_B \tag{142}$$

$$\frac{\partial A_{\text{above}}}{\partial n} - \frac{\partial A_{\text{below}}}{\partial n} = -\mu_0\vec{K} \tag{143}$$

5.5.2 Applications

Multiple expansion:

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(\cos\theta')dl'\\ &= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r^2}\oint r'\cos\theta d\vec{l'}\right.\\ &\left.+\frac{1}{r^3}\oint(r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\vec{l} + \dots\right] &(144)\end{aligned}$$

For the dipole term,

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \tag{145}$$

where $\vec{m}$ is the magnetic dipole moment

$$\vec{m} = I\int d\vec{a} = I\vec{a} \tag{146}$$

5.6 Magnetic Dipole

For an ideal magnetic dipole $\vec{m}$ pointing in the $\hat{z}$ direction,

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi} \tag{147}$$

and

$$\vec{B}_{\text{dip}}(\vec{r}) = \nabla\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{148}$$

In general,

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] \tag{149}$$

$$\vec{\tau} = \vec{m}\times\vec{B} \tag{150}$$

$$\vec{F} = \nabla(\vec{m}\cdot\vec{B}) \tag{151}$$

Note that this is different from the electrical dipole analogue, $\vec{F} = (\vec{p}\cdot\nabla)\vec{E}$ since $\nabla\times\vec{B}\neq 0$ in the magnetostatic case.

6 Magnetization

$$\vec{M} \equiv \text{magentic dipole moment per unit volume} \tag{152}$$

Bound surface current:

$$\vec{K}_b = \vec{M} \times \hat{n} \tag{153}$$

Bound volume current:

$$\vec{J}_b = \nabla \times \vec{M} \tag{154}$$

Such that

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r'})}{\eta} d\tau + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r'})}{\eta} da' \tag{155}$$

Average magnetic field over a sphere with radius R, where $\vec{m}$ is the total dipole moment:

$$\vec{B}_{\text{ave}} = \frac{\mu_0}{4\pi} \frac{2\vec{m}}{R^3} \tag{156}$$

Inside a uniformly magnetized sphere,

$$\vec{B} = \frac{2}{3} \mu_0 \vec{M} \tag{157}$$

6.1 Auxiliary Field $\vec{H}$

Total current is the sum of bound and free currents:

$$\vec{J} = \vec{J}_b + \vec{J}_f \tag{158}$$

$\vec{H}$ is defined as

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \tag{159}$$

such that

$$\nabla \times \vec{H} = \vec{J}_f \tag{160}$$

Ampere's Law:

$$\oint \vec{H} \cdot \vec{dl} = I_{\text{f,enc}} \tag{161}$$

6.2 Magnetostatic Boundary Conditions

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right) \tag{162}$$

$$\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_f \times \hat{n} \tag{163}$$

6.3 Magnetic Susceptibility and Permeability

$$\vec{M} = \chi_m \vec{H} \tag{164}$$

$$\vec{B} = \mu \vec{H}, \mu = \mu_0(1 + \chi_m) \tag{165}$$

6.4 Linear Media Bound and Free Currents

$$\vec{J}_b = \chi_m \vec{J}_f \tag{166}$$

7 Electrodynamics

$$\vec{J} = \sigma \vec{E} \tag{167}$$

$$\vec{J} = \frac{n f \lambda q^2}{2 m v_{\text{thermal}}} \vec{E} \tag{168}$$

where f is the free electrons per molecule and λ is the mean free path.

7.1 Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{169}$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} \tag{170}$$

7.2 Inductance

$$\Phi = MI \tag{171}$$

$$M_{12} = M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{\vec{dl}_1 dl_2}{\eta} \tag{172}$$

$$\epsilon = -L \frac{dI}{dt} \tag{173}$$

$$W = \frac{1}{2} L I^2 \tag{174}$$

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau \tag{175}$$

$$= \frac{1}{2} \oint (\vec{A} \cdot \vec{J}) d\tau \tag{176}$$

$$= \frac{1}{2} \int_V (\vec{A} \cdot \vec{J}) d\tau \quad \text{generalized to volume currents} \tag{177}$$

7.3 Displacement Current

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{178}$$

$$I_d = \epsilon_0 \frac{\partial \Phi_E}{\partial t} \tag{179}$$

7.4 Maxwell-Faraday Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{180}$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot \vec{da} \tag{181}$$

$$= \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} \tag{182}$$

7.5 Maxwell's Equations

7.5.1 Common

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{183}$$

$$\nabla \cdot \vec{B} = 0 \tag{184}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{185}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{186}$$

7.5.2 Fields vs Sources

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{187}$$

$$\nabla \cdot \vec{B} = 0 \tag{188}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{189}$$

$$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \tag{190}$$

7.5.3 Free Charges and Currents

$$\nabla \cdot \vec{D} = \rho_f \tag{191}$$

$$\nabla \cdot \vec{B} = 0 \tag{192}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{193}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \vec{J}_f + \vec{J}_d \tag{194}$$

7.5.4 Boundary Conditions - Linear Media

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \tag{195}$$

$$\vec{E_1}^{\parallel} - \vec{E_2}^{\parallel} = 0 \tag{196}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{197}$$

$$\frac{1}{\mu_1}\vec{B_1}^{\parallel} - \frac{1}{\mu_2}\vec{B_2}^{\parallel} = \vec{K_f} \times \hat{n} \tag{198}$$

where the positive diection for the $\vec{a}$ vector in the derivation is from 2 towards 1.

7.5.5 Boundary Conditions - General

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \tag{199}$$

$$\vec{H_1}^{\parallel} - \vec{H_2}^{\parallel} = \vec{K_f} \times \hat{n} \tag{200}$$

7.5.6 Faraday-induced electric fields

$$\vec{E}(r,t) = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\vec{B}(r',t) \times \hat{\eta}}{\eta^2} d\tau' \tag{201}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{202}$$

8 Conservation Laws

8.1 Poynting's Theorem

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d\tau - \frac{1}{\mu_0}\oint (\vec{E} \times \vec{B}) \cdot d\vec{a} \tag{203}$$

8.1.1 Poynting Vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \tag{204}$$

Unit: Energy per time per area. Therefore,

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S} \cdot d\vec{a} \tag{205}$$

Let U_{mech} be included as:

$$\frac{dW}{dt} = \frac{d}{dt}\int_V U_{mech} d\tau \tag{206}$$

We get the Differential version:

$$\frac{\partial}{\partial t}(U_{mech} + U_{em}) = -\nabla \cdot \vec{S} \tag{207}$$

8.2 Maxwell Stress Tensor

8.2.1 Definition

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{208}$$

where δ_{ij} is the Kronecker Delta, defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{209}$$

$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int_V \vec{S} d\tau \tag{210}$$

$$(\vec{a} \cdot \overleftrightarrow{T})_j = \sum_{i=x,y,z} a_i T_{ij} \tag{211}$$

where T_{ij} is the force per unit area in the ith direction acting on an element of surface in the jth direction. Hence when $i = j$, it is a pressure, and when $i \neq j$ it is a shear.

8.2.2 Total Force on charges in V

$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int_V \vec{S} d\tau \tag{212}$$

8.3 Momentum

8.3.1 Density of EM momentum

$$\vec{\rho}_{em} = \mu_0\epsilon_0\vec{S} \tag{213}$$

8.3.2 Conservation of momentum

$$\frac{\partial}{\partial t}(\vec{\rho}_{mech} + \vec{\rho}_{em}) = \nabla \cdot \overleftrightarrow{T} \tag{214}$$

Hence we see that $\overleftrightarrow{T}$ is the momentum flux density. T_{ij} is hence the momentum in the i direction crossing a surface in the j direction, per unit area, per unit time.

8.3.3 Density of EM Angular momentum

$$l_{em} = \epsilon_0[\vec{r} \times (\vec{E} \times \vec{B})] = \vec{r} \times \vec{\rho}_{em} \tag{215}$$

9 Waves

9.1 Definition

$$f(z,t) = g(z - vt) + h(z + vt) \tag{216}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{in one dimension} \tag{217}$$

$$k = \frac{2\pi}{\lambda} \tag{218}$$

$$\omega = 2\pi f \tag{219}$$

9.2 Boundary Conditions

$$f(0^-,t) = f(0^+,t) \tag{220}$$

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \text{without knot} \tag{221}$$

$$T\left(\left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-}\right) = m\frac{\partial^2 f}{\partial t^2} \text{with knot} \tag{222}$$

9.3 Reflection and Transmission

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \tag{223}$$
$$k_1(\tilde{A}_1 - \tilde{A}_R) = k_2\tilde{A}_T \quad \text{without knot} \tag{224}$$
$$A_R = \left(\frac{v_1 - v_2}{v_1 + v_2}\right) A_I \tag{225}$$
$$A_T = \left(\frac{2v_2}{v_1 + v_2}\right) A_I \tag{226}$$

9.4 Electromagnetic Waves

$$\tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{E}_0) \tag{227}$$
$$B_0 = \frac{E_0}{c} \tag{228}$$
$$\tilde{E}(\vec{r}, t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\hat{n} \tag{229}$$
$$\tilde{B}(\vec{r}, t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}(\hat{k} \times \hat{n}) = \frac{1}{c}\hat{k} \times \tilde{E} \tag{230}$$

9.5 Real Waves

$$\vec{E}(\vec{r}, t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{n} \tag{231}$$
$$\vec{B}(\vec{r}, t) = \frac{1}{c} E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k} \times \hat{n}) \tag{232}$$

9.6 Energy and Momentum

$$\vec{S} = cu\hat{k} \tag{233}$$
$$\vec{\wp}_{\text{density}} = \frac{\vec{S}}{c^2}, \text{the momentum density} \tag{234}$$

For monochromatic plane waves,

$$\vec{\wp}_{\text{density}} = \frac{u}{c}\hat{k} \tag{235}$$

where u is the energy density.

9.7 Averages

Note that the average of $\cos^2\theta = \frac{1}{2}$.

$$< u >= \frac{1}{2}\epsilon_0 E_0^2 \tag{236}$$
$$< \vec{S} >= \frac{1}{2}c\epsilon_0 E_0^2\hat{k} = I\hat{k} \tag{237}$$

where $I = \frac{1}{2}c\epsilon_0 E_0^2$ is the intensity, average power per unit area transported by the EM wave.

$$< \vec{\wp}_{\text{density}} >= \frac{1}{2c}\epsilon_0 E_0^2\hat{k} \tag{238}$$

9.8 Radiation Pressure

$$P = \begin{cases} \frac{I}{c}, & \text{absorption} \\ \frac{2I}{c}, & \text{perfect reflector} \end{cases} \tag{239}$$

9.9 Reflection and Transmission Amplitudes

$$E_{O_R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{O_I} \tag{240}$$
$$E_{O_T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{O_I} \tag{241}$$

for normal incidence.

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \tag{242}$$
$$T = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{243}$$
$$R + T = 1 \tag{244}$$

9.10 Oblique Incidence: Fresnel Equations

$$\tilde{E}_{O_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{O_I} \tag{245}$$
$$\tilde{E}_{O_T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{O_I} \tag{246}$$
$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \tag{247}$$
$$\alpha = \frac{\sqrt{1 - \left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I} \tag{248}$$
$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{249}$$

9.10.1 Brewster's Angle

$$\sin^2\theta_B = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2} \tag{250}$$

when $\mu_1 \approx \mu_2$, $\beta \approx \frac{n_2}{n_1}$, $\sin^2\theta_B \approx \frac{\beta^2}{1+\beta^2}$ hence,

$$\tan\theta_B \approx \frac{n_2}{n_1} \tag{251}$$

9.10.2 R and T Coefficients

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \tag{252}$$
$$T = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2 \tag{253}$$

9.11 Absorption

Let $J_f = \sigma\vec{E}$

9.11.1 Modified Wave Equations

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \tag{254}$$
$$\nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t} \tag{255}$$

with solutions:

$$\vec{E} = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)} \tag{256}$$
$$\vec{B} = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)} \tag{257}$$

But now $\tilde{k}$ is complex:

$$\tilde{k} = k_r + ik_i \tag{258}$$
$$k_r = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1} \tag{259}$$
$$k_i = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}-1} \tag{260}$$

$\tilde{k}$ has to satisfy the following: (by plugging Eq.256 and Eq.257 into Eq.254 and Eq.255)

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{261}$$

Re-expressing solutions,

$$\tilde{E} = \tilde{E}_0 e^{-k_i z} e^{i(k_r z-\omega t)} \tag{262}$$
$$\tilde{B} = \tilde{B}_0 e^{-k_i z} e^{i(k_r z-\omega t)} \tag{263}$$

9.11.2 Skin Depth

$$d \equiv \frac{1}{k_i} \tag{264}$$

9.11.3 Phase Difference

Let

$$\tilde{k} = Ke^{i\phi} \tag{265}$$

with

$$K = \omega\sqrt{\epsilon\mu\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}} \quad \text{and} \tag{266}$$
$$\phi = \tan^{-1}\frac{k_i}{k_r} \tag{267}$$

Also, the magnetic field lags behind the electric field:

$$\delta_B - \delta_E = \phi \tag{268}$$

Hence,

$$\vec{E}(z,t) = E_0 e^{-k_i z}\cos(k_r z - \omega t + \delta_E)\hat{x} \tag{269}$$
$$\vec{B}(z,t) = B_0 e^{-k_i z}\cos(k_r z - \omega t + \delta_E + \phi)\hat{y} \tag{270}$$

9.12 Reflection at conducting surface

$$\tilde{E}_{O_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{O_I} \tag{271}$$
$$\tilde{E}_{O_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{O_I} \tag{272}$$

with

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2 \tag{273}$$

9.13 Dispersion

$$v = \frac{\omega}{k} \tag{274}$$
$$v_g = \frac{d\omega}{dk} \tag{275}$$

Energy carried travels at the group velocity.

9.13.1 Damped Harmonic Oscillator

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos\omega t \tag{276}$$

Hence the dipole moment

$$\tilde{p}(t) = \frac{q^2/m}{\omega_0^2-\omega^2-i\gamma\omega}E_0 e^{-i\omega t} \tag{277}$$

The long range dipole moment:

$$\tilde{P} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}\right)\tilde{E} \tag{278}$$

with N molcules per unit volume, f_j electrons with frequency ω_j and damping γ_j.
We hence have the complex susceptibility:

$$\tilde{P} = \epsilon_0\tilde{\chi}_E\tilde{E} \tag{279}$$

and complex dielectric constant:

$$\tilde{\epsilon_r} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma\omega} \tag{280}$$

9.13.2 Dispersive Wave Equation

$$\nabla^2\tilde{E} = \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{E}}{\partial t^2} \tag{281}$$

with solution

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)} \tag{282}$$
$$\tilde{k} = \omega\sqrt{\tilde{\epsilon}\mu_0} = \sqrt{\tilde{\epsilon_r}}\frac{\omega}{c} \tag{283}$$

Hence,

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-k_i z} e^{i(k_r z-\omega t)} \tag{284}$$

Since the intensity is proportional to the square of the amplitude, the intensity falls off with $\alpha = 2k_i$.

9.14 Dilute Gases

Under binomial expansion,

$$\tilde{k} \approx \frac{\omega}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}\right) \tag{285}$$
$$n = \frac{ck_r}{\omega} \approx 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega} \tag{286}$$
$$\alpha = 2k_i \approx \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2} \tag{287}$$

Away from resonances,

$$n = 1 + \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j}{\omega_j^2 - \omega^2} \tag{288}$$

For transparent materials, $\omega << \omega_j$, hence we can express n in the form:

$$n \approx 1 + A\left(1 + \frac{B}{\lambda^2}\right) \quad \text{(Cauchy's formula)} \tag{289}$$

$$A = \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2} \quad \text{(Coefficient of refraction)} \tag{290}$$

$$B = 4\pi^2 c^2 \frac{\sum_j \frac{f_j}{\omega_j^4}}{\sum_j \frac{f_j}{\omega_j^2}} \quad \text{(Coefficient of dispersion)} \tag{291}$$

9.15 Wave Guides

9.15.1 Boundary Conditions

$$\vec{E}^{\parallel} = 0 \tag{292}$$

$$\vec{B}^{\perp} = 0 \tag{293}$$

inside the wave guide (i.e. at the inner wall). Magnetic field wil remain at zero if it had started out at zero (since $\vec{E} = 0$ in a perfect conductor, and $-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = 0$.

9.15.2 Uncoupled Equations

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right) - k^2\right] E_z = 0 \tag{294}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right) - k^2\right] B_z = 0 \tag{295}$$

9.15.3 Types

- TE: Transverse Electric

$$E_z = 0 \tag{296}$$

- TM: Transverse Magnetic

$$B_z = 0 \tag{297}$$

- TEM: Transverse Electric Magnetic

$$E_z = B_z = 0 \tag{298}$$

TEM cannot occur in a hollow wave guide. By Gauss' Law and Faraday's Law, can prove that $\tilde{E}_0$ has zero divergence and zero curl. Hence is a gradient of scalar potential, and cannot have local extremum. If boundary condition is equipotential, then potential is constant everywhere=zero electric field.

9.15.4 Rectangular Waveguide

For TE_{mn},

$$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{299}$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \tag{300}$$

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{301}$$

where ω_{mn} is the cutoff frequency.

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} \tag{302}$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c \tag{303}$$

9.15.5 Coaxial Line

$$\vec{E} = \frac{A\cos(kz - \omega t)}{s}\hat{s} \tag{304}$$

$$\vec{B} = \frac{A\cos(kz - \omega t)}{cs}\hat{\phi} \tag{305}$$

10 Potentials and Fields

10.1 Link to Maxwell's Equations

Combining and expressing Maxwell's Equations in potential form (i.e. $V, \vec{A}$),

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \tag{306}$$

$$\left(\nabla^2 \vec{A} - \mu_0\epsilon_0\frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \tag{307}$$

Equivalently,

$$\Box^2 V + \frac{\partial L}{\partial t} = -\frac{\rho}{\epsilon_0} \tag{308}$$

$$\Box^2 \vec{A} - \nabla L = -\mu_0\vec{J} \tag{309}$$

with

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \tag{310}$$

$$L \equiv \nabla \cdot \vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{311}$$

$\Box^2$ is called the d'Alembertian.

10.2 Gauge Transformations

Gauge freedom:

$$\vec{A}' = \vec{A} + \nabla\lambda \tag{312}$$

$$V' = V - \frac{\partial\lambda}{\partial t} \tag{313}$$

10.2.1 Coulomb Gauge

$$\nabla \cdot \vec{A} = 0 \tag{314}$$

10.2.2 Lorentz Gauge

$$\nabla \cdot \vec{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{315}$$

Under the Lorentz Gauge,

$$\Box^2 V = -\frac{\rho}{\epsilon_0} \tag{316}$$

$$\Box^2 \vec{A} = -\mu_0\vec{J} \tag{317}$$

10.3 Retarded Potentials

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'},t_r)}{\eta}dt' \tag{318}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'},t_r)}{\eta}dt' \tag{319}$$

with the retarded time t_r defined as

$$t_r \equiv t - \frac{\eta}{c} \tag{320}$$

10.3.1 Jefimenko's Equations

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\vec{r'},t_r)}{\eta^2}\hat{\eta} + \frac{\dot{\rho}(\vec{r'},t_r)}{c\eta}\hat{\eta} - \frac{\dot{\vec{J}}(\vec{r'},t_r)}{c^2\eta}\right]d\tau' \tag{321}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \left[\frac{\vec{J}(\vec{r'},t_r)}{\eta^2} + \frac{\dot{\vec{J}}(\vec{r'},t_r)}{c\eta}\right] \times \hat{\eta}d\tau' \tag{322}$$

10.3.2 Liénard-Wiechert Potentials

Retarded potentials of a point charge q moving along path $\vec{w}(t)$. Geometric factor:

$$\int \rho(\vec{r'},t_r)d\tau' = \frac{q}{1-\frac{\hat{\eta}\cdot\vec{v}}{c}} \tag{323}$$

Retarded time: solve for t_r from the following implicit equation:

$$|\vec{r} - \vec{v}t_r| = c(t-t_r) \tag{324}$$

Expressing $\vec{A}$ and V,

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\eta c - \vec{\eta}\cdot\vec{v}} \tag{325}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\eta c - \vec{\eta}\cdot\vec{v}} = \frac{\vec{v}}{c^2}V(\vec{r},t) \tag{326}$$

where $\vec{\eta}$ is the vector from the retarded position to field point $\vec{r}$:

$$\vec{\eta} = \vec{r} - \vec{w}(t_r) \tag{327}$$

10.3.3 Problem solving with retarded potentials

1. $t_r = t - \frac{|\vec{r}-\vec{w}(t_r)|}{c}$
2. Isolate t_r in terms of x, y, z, t.
3. $V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\eta c - \vec{\eta}\cdot\vec{v}}$ with $\vec{v} = \dot{\vec{w}}(t_r)$
4. Solve for $\eta c - \vec{\eta}\cdot\vec{v}$ in terms of x, y, z, t.
5. $\vec{A} = \frac{\vec{v}}{c^2}V$

10.3.4 Fields of a moving point charge

From $\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \nabla\times\vec{A}$,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\eta}{(\vec{\eta}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\eta}\times(\vec{u}\times\vec{a})\right] \tag{328}$$

$$\vec{B} = \frac{1}{c}\hat{\eta}\times\vec{E} \tag{329}$$

$$\vec{u} \equiv c\hat{\eta} - \vec{v} \tag{330}$$

Note that $\vec{E}$ =Velocity field (falls off $\propto \frac{1}{r^2}$)+Acceleration Field (falls off $\propto \frac{1}{r}$). Hence, at large distances, only the acceleration field is dominant, and this contributes to radiation.

11 Radiation

11.1 Electric Dipole Radiation

Given $\vec{p}(t) = p_0\cos(\omega t)\hat{z}$,

11.1.1 Approximations

$$d << \frac{c}{\omega} = \frac{\lambda}{2\pi} \tag{331}$$

$$r >> \frac{c}{\omega} = \frac{\lambda}{2\pi} \tag{332}$$

$$\tag{333}$$

Implicitly, $r >> d$.

11.1.2 Potentials

$$V = \frac{-\omega}{4\pi\epsilon_0 c}\frac{\vec{p_0}\cdot\hat{r}}{r}\sin\left[\omega(t-\frac{r}{c})\right] \tag{334}$$

$$\vec{A} = \frac{-\mu_0\omega}{4\pi}\frac{\vec{p_0}}{r}\sin\left[\omega(t-\frac{r}{c})\right] \tag{335}$$

11.1.3 Fields and Derivatives

$$\vec{E} = \frac{\mu_0\omega^2}{4\pi}\frac{\hat{r}\times(\vec{p_0}\times\hat{r})}{r}\cos\left[\omega(t-\frac{r}{c})]\right] \tag{336}$$

$$\vec{B} = \frac{-\mu_0\omega^2}{4\pi c}\frac{\vec{p_0}\times\hat{r}}{r}\cos\left[\omega(t-\frac{r}{c})\right] \tag{337}$$

$$<\vec{S}> = \frac{\mu_0\omega^4}{32\pi^2 c}\frac{(\vec{p_0}\times\hat{r})^2}{r^2}\hat{r} \tag{338}$$

$$<P> = \frac{\mu_0 p_0^2\omega^4}{12\pi c} \tag{339}$$

11.2 Magnetic Dipole Radiation

Given $\vec{m} = m_0\cos(\omega t)\hat{z}$,

11.2.1 Potentials

Since the loop is uncharged,

$$V = 0 \tag{340}$$

Also,

$$\vec{A} = \frac{-\mu_0 m_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin\left[\omega(t-\frac{r}{c})\right]\hat{\phi} \tag{341}$$

11.2.2 Fields and Derivatives

$$\vec{E} = \frac{\mu_0 m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega(t-\frac{r}{c})\right]\hat{\phi} \tag{342}$$

$$\vec{B} = \frac{-\mu_0 m_0\omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\left[\omega(t-\frac{r}{c})\right]\hat{\theta} \tag{343}$$

$$<\vec{S}> = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{r} \tag{344}$$

$$<P> = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3} \tag{345}$$

11.3 Radiation from an Arbitrary Source

11.3.1 Approximations

1. $r' << r$
2. $r' << \frac{c}{|\ddot{\rho}/\rho|}$ and higher powers.
   For 1st order Taylor expansion of $\rho(\vec{r'}, t - \frac{\eta}{c})$ with respect to t.
3. Discard $\frac{1}{r^2}$ terms in $\vec{E}, \vec{B}$ Only $\frac{1}{r}$ terms contribute to radiation anyway.

11.3.2 Fields

$$\vec{E} = \frac{\mu_0}{4\pi r}\left[\hat{r} \times (\hat{r} \times \ddot{\vec{p}})\right] \tag{346}$$

$$\vec{B} = \frac{-\mu_0}{4\pi rc}[\hat{r} \times \ddot{\vec{p}}] \tag{347}$$

Note that $\ddot{\vec{p}}$ is evaluated at time $t_0 = t - \frac{r}{c}$.

11.3.3 Larmor Formula

$$P = \frac{\mu_0 a^2 q^2}{6\pi c} \tag{348}$$

11.3.4 Liénard's Generalization

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v} \times \vec{a}}{c}\right|\right) \tag{349}$$

where γ is the Lorentz factor. Reduces to the Larmor formula when $v << c$.

11.4 Radiation Reaction Force

The Abraham-Lorentz formula:

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}} \tag{350}$$

12 Electrodynamics and Relativity

12.1 Lorentz Transformation

$$\bar{x} = \gamma(x - vt) \tag{351}$$

$$\bar{y} = y \tag{352}$$

$$\bar{z} = z \tag{353}$$

$$\bar{t} = \gamma(t - \frac{v}{c^2}x) \tag{354}$$

12.2 Four-vectors

A set of four components that transform in the same manner as (x^0, x^1, x^2, x^3) under Lorentz transformations.

12.2.1 Lorentz Transformation in 4-vector form

$$\bar{a}^0 = \gamma(a^0 - \beta a^1) \tag{355}$$

$$\bar{a}^1 = \gamma(a^1 - \beta a^0) \tag{356}$$

$$\bar{a}^2 = a^2 \tag{357}$$

$$\bar{a}^3 = a^3 \tag{358}$$

12.2.2 Dot Product

Covariant vector a_μ:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3) \tag{359}$$

Contravariant vector a^μ:

$$a_0 = -a^0 \quad \text{for temporal indices} \tag{360}$$

$$a_{1,2,3} = a^{1,2,3} \quad \text{for spatial indices} \tag{361}$$

Hence the scalar product is:

$$a_\mu b^\mu = \sum_{\mu=0}^{3} a_\mu b^\mu \tag{362}$$

$a_\mu b^\mu$ is an implied summation.

$$a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{363}$$

12.3 Intervals

$$I = (\triangle x)_\mu (\triangle x)^\mu = -c^2 t^2 + d^2 \tag{364}$$

If $I < 0$, interval is timelike. An appropriate frame can be chosen so that the events occur at the same point.
If $I > 0$, interval is spacelike. An appropriate frame can be chosen so that the events occur at the same time.
If $I = 0$, interval is lightlike.

12.4 Relativistic Mechanics and Dynamics

12.4.1 Proper time

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}}dt = \frac{dt}{\gamma} \tag{365}$$

12.4.2 Proper velocity

Ordinary velocity is defined as such:

$$\vec{u} = \frac{d\vec{l}}{dt} \tag{366}$$

But proper velocity is defined with respect to the proper time τ.

$$\vec{\eta} = \frac{d\vec{l}}{d\tau} \tag{367}$$

$$\vec{\eta} = \frac{\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\vec{u} \tag{368}$$

$$\eta^\mu = \frac{dx^\mu}{d\tau} \tag{369}$$

$$\eta^0 = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma c \tag{370}$$

12.4.3 Relativistic Momentum

$$\vec{p} = m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{371}$$

$$p^\mu = m\eta^\mu \tag{372}$$

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{E}{c} \tag{373}$$

$$p^\mu p_\mu = -m^2 c^2 \tag{374}$$

12.4.4 Relativistic Energy

$$E^2 - p^2c^2 = m^2c^4 \tag{375}$$

12.4.5 Relativistic Force Transform

$$\bar{F}_x = \frac{F_x - \beta(\vec{u}\cdot\vec{F})/c}{1-\beta u_x/c} \tag{376}$$

$$\bar{F}_y = \frac{F_y}{\gamma(1-\beta u_x/c)} \tag{377}$$

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)} \tag{378}$$

12.4.6 Minkowski Force - the "proper" force

$$K^\mu = \frac{dp^\mu}{d\tau} \tag{379}$$

$$\vec{K} = \frac{\vec{F}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{380}$$

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} \tag{381}$$

12.5 Transformation of $\vec{E}, \vec{B}$ fields.

$$\bar{E}_x = E_x \tag{382}$$

$$\bar{E}_y = \gamma(E_y - vB_z) \tag{383}$$

$$\bar{E}_z = \gamma(E_z + vB_y) \tag{384}$$

$$\bar{B}_x = B_x \tag{385}$$

$$\bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \tag{386}$$

$$\bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \tag{387}$$

$$\tag{388}$$

If $\vec{B} = 0$ in a frame, then for all other frames,

$$\vec{\bar{B}} = -\frac{1}{c^2}(\vec{v}\times\vec{E}) \tag{389}$$

at that point.

12.5.1 Invariant Quantities

$$(\vec{E}\cdot\vec{B}) \quad \text{is invariant} \tag{390}$$

$$E^2 - c^2B^2 \quad \text{is invariant} \tag{391}$$

12.6 Field Tensor

$$F^{\mu v} = \begin{Bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{Bmatrix} \tag{392}$$

12.6.1 Transformation of a second-rank tensor

$$E^{uv} = \Lambda^\mu_\lambda \Lambda^v_\sigma t^{\lambda\sigma} \tag{393}$$

with implied summation across all possible (non-zer) combinations of λ, σ. t^{uv} is the entry in the uth row an vth column (indices range from 0...3).

12.6.2 Symmetry

Field tensor is an antisymmetric second rank tensor. Antisymmetric:

$$t^{\mu v} = -t^{v\mu} \tag{394}$$

Second rank: Array is 2 dimensional (i.e. a matrix), since you need 2 indices to label a component of the array.

12.6.3 Dual Tensor

An alternative to the Field Tensor:

$$G^{uv} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & \frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{Bmatrix} \tag{395}$$

12.6.4 Covariant tensors and contravariant tensors

When lowering an index to make it covariant, change sign if any of the indices is zero (if both are zero, the negatives cancel out).

$$\begin{aligned} F^{\mu v}F_{\mu v} =& F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02}... \\ & - F^{30}F^{30} + F^{11}F^{11} + F^{12}F^{12}... + F^{33}F^{33} \end{aligned} \tag{396}$$

12.6.5 Invariants

$$F^{\mu v}F_{\mu v} = 2\left(B^2 - \frac{E^2}{c^2}\right) \tag{397}$$

$$G^{\mu v}G_{\mu v} = 2\left(\frac{E^2}{c^2} - B^2\right) \tag{398}$$

$$F^{\mu v}G_{\mu v} = -\frac{4}{c}(\vec{E}\cdot\vec{B}) \tag{399}$$

12.7 Electrodynamics in Tensor Notation

12.7.1 Current density four-vector

$$J^\mu = (c\rho, J_x, J_y, J_z) \tag{400}$$

$$\rho = \frac{\rho_0}{\sqrt{1-\frac{u^2}{c^2}}} \tag{401}$$

where ρ_0 is the proper charge density.

12.7.2 Continuity equation

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \tag{402}$$

with implied summation across μ.

12.7.3 Maxwell's Equations

$$\frac{\partial F^{\mu v}}{\partial x^v} = \mu_0 J^\mu \tag{403}$$

With $\mu = 0$, becomes $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$.
With $\mu = 1,2,3$, becomes $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\frac{\partial G^{\mu v}}{\partial x^v} = 0 \tag{404}$$

With $\mu = 0$, becomes $\nabla\cdot\vec{B} = 0$.
With $\mu = 1,2,3$, becomes $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

12.7.4 Minkowski Force on charge

$$K^\mu = q\eta_v F^{\mu v} \tag{405}$$

Wih $\mu = 0$, becomes $\frac{dW}{dt} = q(\vec{u} \cdot \vec{E})$
With $\mu = 1, 2, 3$, becomes $\vec{K} = \frac{q}{\sqrt{1-\frac{u^2}{c^2}}}[\vec{E} + (\vec{u} \times \vec{B})]$.

12.7.5 Relativistic Potentials

$$A^\mu = (v/c, A_x, A_y, A_z) \tag{406}$$

$$F^{\mu v} = \frac{\partial A^v}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_v} \tag{407}$$

implies $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$.

12.7.6 Lorentz Gauge

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \implies \frac{\partial A^\mu}{\partial x^\mu} = 0 \tag{408}$$

Under the Lorentz Gauge,

$$\Box^2 A^\mu = -\mu_0 J^\mu \tag{409}$$

$$\Box^2 = \frac{\partial}{\partial x_v}\frac{\partial}{\partial x^v} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{410}$$

12.8 Four-dimensional Gradient

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu \quad \text{covariant gradient} \tag{411}$$

$$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu \quad \text{contravariant gradient} \tag{412}$$