Fee-Six v14
Mathematics
Differential Equations
y'+p(x)y=q(x)
Integrating Factor $e^{P(x)}$, P(x) is the antiderivative of p(x)

y''+py'+qy=F(x)
Solve complement by guess $e^{\lambda t}$ to obtain $y_c=c_1e^{\lambda 1t}+c_2e^{\lambda 2t}$
Solve particular by observation
Undetermined polynomial for polynomial
Exponential and sinusoidal sub guesses respectively

Bernoulli DE
$y'+py=qy^n$
Define $z=y^{1-n}$, $dz=(1-n)y^{-n}y'dy$
Multiply DE by $(1-n)y^{-n}$

Taylor Expansion
$f(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2$
$f(x+a)=f(a)+f'(a)x+\frac{1}{2}f''(a)x^2$
Binomial Expansion
$(1+x)^n=1+nx+\frac{1}{2}n(n-1)x^2$

Hyperbolic Identities
$\cosh x=\frac{e^x+e^{-x}}{2}$, $\sinh x=\frac{e^x-e^{-x}}{2}$, $\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$
$\cosh^2x-\sinh^2x=1$
$\cosh x+\sinh x=e^x$, $\cosh x-\sinh x=e^{-x}$
$\sinh 2x=2\cosh x\sinh x$
$\cosh 2x=2\cosh^2x-1=1+2\sinh^2x=\cosh^2x+\sinh^2x$

Trigonometric to Complex (and v.v.)
$e^{i\theta}+e^{-i\theta}=2\cos\theta$
$e^{i\theta}-e^{-i\theta}=2\sin\theta$
$\cosh(i\theta)=\cos\theta$
$\sinh(i\theta)=\sin\theta$
$\tanh(i\theta)=i\tan\theta$

Triple Angle Formulae
$\sin(3x)=3\sin(x)-4\sin^3(x)$
$\cos(3x)=4\cos^3(x)-3\cos(x)$

Factor Formulae
$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Sinusoidal Conversions
$\sin(\omega t)=\cos(90-\omega t)$
$\cos(\omega t)=\sin(90-\omega t)$
$\sin(\omega t\pm 180)=-\sin(\omega t)$
$\cos(\omega t\pm 180)=-\cos(\omega t)$
$\sin(\omega t\pm 90)=\pm\cos(\omega t)$
$\cos(\omega t\pm 90)=\mp\sin(\omega t)$

Stirling's Approximation
ln N! ~ N ln N - N
$N!\sim\sqrt{2\pi N}\left(\frac{N}{e}\right)^N$

Coordinate Systems
Cartesian (x,y,z)
Cartesian Volume: dxdydz
Spherical (r,θ,φ)
Spherical Line: rdθ
Spherical Area: $r^2\sin\theta d\theta d\varphi$
Spherical Volume: $r^2\sin\theta d\theta d\varphi dr$
Spherical Limits: $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$
Cylindrical (r,φ,z)
Cylindrical Volume: rdφdzdr
Cylindrical Limits: $\varphi\in[0,2\pi]$

Spherical to Cartesian conversion
x=r sin θ cos φ
y= r sin θ sin φ
z= r cos θ

Gradient Function
Cartesian:
$$\nabla A(x,y,z)=\hat{x}\frac{\partial A}{\partial x}+\hat{y}\frac{\partial A}{\partial y}+\hat{z}\frac{\partial A}{\partial z}$$
Spherical:
$$\nabla A(r,\theta,\phi)=\hat{r}\frac{\partial A}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial A}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial A}{\partial\phi}$$
Cylindrical:
$$\nabla A(r,\theta,z)=\hat{r}\frac{\partial A}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial A}{\partial\theta}+\hat{z}\frac{\partial A}{\partial z}$$

Divergence
$$\nabla\cdot A(x,y,z)=\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial z}A_z$$
$$\nabla\cdot A(r,\phi,z)=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial}{\partial\phi}A_\phi+\frac{\partial}{\partial z}A_z$$
$$\nabla\cdot A(r,\theta,\phi)=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

Curl
$$\nabla\times\vec{A}(x,y,z)=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$
$$\nabla\times\vec{A}(r,\phi,z)=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$$

Divergence Theorem
$$\iiint(\vec{\nabla}\cdot\vec{A})dV=\oiint_S\vec{A}\cdot d\vec{S}$$

Curl Theorem/Stokes' Theorem
$$\iint(\vec{\nabla}\times\vec{A})dS=\oint_C\vec{A}\cdot d\vec{l}$$

Vector Identities
$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$
$\vec{\nabla}\times\left(\vec{\nabla}A\right)=0$
$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$
$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}-(\vec{A}\times\vec{C})\cdot\vec{B}$
Total Derivative
$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial t}dt$
$df=\nabla f\cdot dl$

Least Square Fitting
Let y=a+bx
$$a=\frac{\sum y\sum x^2-\sum x\sum xy}{N\sum x^2-\left(\sum x\right)^2}$$
$$b=\frac{N\sum xy-\sum x\sum y}{N\sum x^2-\left(\sum x\right)^2}$$
$$\Delta a=\sqrt{\frac{\sum y^2-a\sum y-b\sum xy}{N(N-2)}}$$
$$\Delta b=\frac{\Delta a}{\sqrt{\left(\sum x^2\right)-\frac{\left(\sum x\right)^2}{N}}}$$

Error Analysis
For $A(x_1, x_2, x_3...)$:
$\Delta A=\frac{\partial A}{\partial x_1}\Delta x_1+\frac{\partial A}{\partial x_2}\Delta x_2+\frac{\partial A}{\partial x_3}\Delta x_3...$

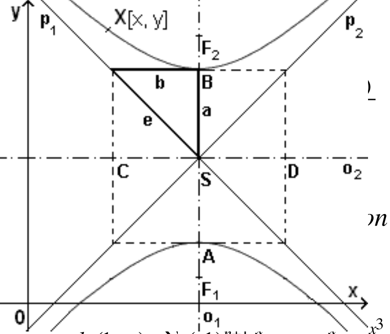
Arc Length
$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$
$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$
$ds=\sqrt{r(\theta)^2+\left(\frac{dr(\theta)}{d\theta}\right)^2}d\theta$

Volume of revolution
Around x-axis: $V=\pi\int y^2dx$
Around y-axis: $V=\pi\int x^2dy$

Volume of Ellipsoid
$V=\frac{4}{3}\pi ab^2$
If total volume need to be equal to the volume of a sphere, then
$a=R(1+\varepsilon)$
$b=\frac{R}{\sqrt{1+\varepsilon}}$
Where ε is the parameter of deformation

Surface Area of Revolution
Around x-axis
$A=\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$

Series Summation



$ln(1+x)=\sum_{n=1} (-1)^{n+1} \frac{x^n}{n} = x-\frac{x^2}{2}+\frac{x^3}{3}...$

$sin\ x = x - \frac{x^3}{3!}+\frac{x^5}{5!}$

$cos\ x = 1 - \frac{x^2}{2!}+\frac{x^4}{4!} - ....$

$tan\ x = x + \frac{x^3}{3}+\frac{2x^5}{15}..., |x|<\frac{\pi}{2}$

Useful Summation Identities

$$\sum_{k=1}^{N} k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{N} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^{N} k^3 = \left(\sum_{k=1}^{N} k\right)^2$$

Finding Determinants

1. Gaussian Algorithm: Form an upper triangular matrix (bottom triangle entries are zeros). The determinant of an upper triangular matrix is the product of the entries on the main diagonal
2. Laplace Expansion: Pick a row or column with many zeros. Sum:

   $$\sum_{i,j} (-1)^{i+j}\, b_{i,j} |M_{i,j}|$$

   Where $b_{i,j}$ is the entry on the ith row and jth column, $M_{i,j}$ is the matrix with the ith row and jth column removed

Cross Products

$a\times(b+c)=a\times b+a\times c$

Parallelepiped volume: $V=|(a\times b)\cdot c|$

Triple Product

(a x b) • c = (c x a) • b = (b x c) • a

a x (b x c) = b(a • c) - c(a • b)

Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

a= semimajor axis

b= semi minor axis

Hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

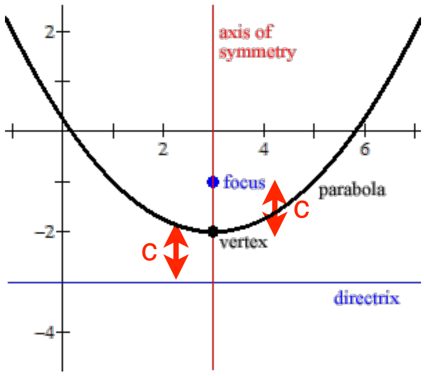
a= vertex distance from origin

b= distance of vertex to asymptotic line

Parabola

$x^2=4cy$

c=distance of focus from vertex or distance of vertex from directrix



Each point on the parabola is equidistant from the focus and perpendicular distance to directrix

Equation of plane

$A(x-x_o)+B(y-y_o)+C(z-z_o)=0$

where the normal vector is given by N=A[x]+B[y]+C[z] and (x_o,y_o,z_o) is a point on the plane

Equation of line

$(y-y_o)=m(x-x_o)$

Regular Polygon

Sum of internal angles: (n-2)(180°)

Radius (distance of vertice to centre):

$r=\frac{s}{2\sin\left(\frac{\pi}{n}\right)}$, s=length of one side

Apothem: Perpendicular distance to centre

$a=\frac{s}{2\tan\left(\frac{\pi}{n}\right)}$

$A=\frac{1}{4}ns^2\cot\left(\frac{\pi}{n}\right)=na^2\tan\left(\frac{\pi}{n}\right)=\frac{1}{2}nsa$

Mean Value

Given $f(x)=e^{-\frac{x}{a}}$,

$<f(x)> = a$

Partial Fractions

$$\frac{p(x)}{(x+a)(x+b)^n}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}+...+\frac{Z}{(x+b)^n}$$

$$\frac{p(x)}{(x+a)(x^2+bx+c)}=\frac{A}{x+a}+\frac{Bx+C}{x^2+bx+c}$$

Mechanics

Time constant

$e^{-t/T}$, T=time constant

Conservative Field

$\oint A\bullet dl=0$

$\vec{\nabla}\times\vec{A}=0$

Polar Coordinates Vectors

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\theta}$$

A central force only depends on distance r, no θ acceleration. Hence the derivative=0, and $r^2\omega=L/m$=constant.

COAM holds for all central forces.

Potential Energy

W=F•r=∫F•dr

$F=-\frac{dU}{dx},\ \Delta U=-\int_{initial\ x}^{final\ x} F\cdot dx$

$U=-\int_{r}^{\infty} F\cdot dr=\int_{\infty}^{r} F\cdot dr$

Potential

$$\Delta V=\frac{\Delta U}{m}$$

Rocket Propulsion

$P_f-P_i=F_{net}dt$

$P_i=mv$

$P_f=(m-dm)(v+dv)+(-u+v)(dm)$

$\Delta V= -v_e ln\left(\frac{m_f}{m_i}\right)$

$F= -v_e\frac{dM}{dt}$

Center of mass

$$r_{cm}=\frac{1}{M}\sum_{i=1}^{N} m_i r_i=\frac{1}{M}\int_{x_1}^{x_2} rdm$$

$$v_{cm}=\frac{1}{M}\sum_{i=1}^{N} m_i v_i=\frac{1}{M}\int_{x_1}^{x_2} vdm$$

Eccentricity

e=c/a

$a^2=b^2+c^2$

a=semimajor axis

$-\frac{GMm}{2a}$=E, $a=-\frac{GMm}{2E}$

b=semiminor axis

c=focus distance from center

$e=(r_a-r_p)/(r_a+r_p)$

$e=1-\frac{2}{\left(\frac{r_a}{r_p}\right)+1}$

$e=\sqrt{1+\frac{2EL^2}{G^2M^2m^3}}$

Generalized eccentricity for inverse-square forces $F= -k/r^2$

$e=\sqrt{1+\frac{2EL^2}{mk^2}}$

r_a=apoapsis distance=a+c

Apoapsis distance = a(1+ε)

r_p=periapsis distance=a-c

Periapsis distance = a(1-ε)

Equation of orbit

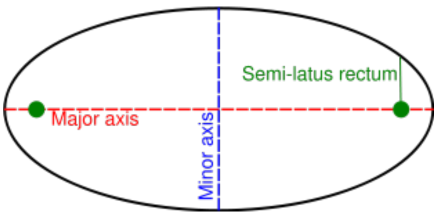
$r(\theta)=\frac{l}{1-\varepsilon\cos\theta}$

I=semi-latus rectum

ε=eccentricity

θ=Angle from semimajor axis

Semi-latus rectum:



I for circle is r

When θ=0, r_{max}=I/(1-ε) = a+c

When θ=π, r_{min}=I/(1+ε) = a-c

Calculation of Semi-latus rectum

I= a | 1- $ε^2$ |, a=semimajor axis

Since ε=c/a, I=b^2/a

$I=\frac{L^2}{GMm^2}=\frac{v^2r^2}{GM}$

Residual Velocity

v_∞, only if e≥1

Impact Parameter

Perpendicular distance from focus at infinity

By POCOAM: $mv_or_o=mv_\infty b$

b=impact parameter

Fictitious forces

Centrifugal Force

$\vec{F}_{cen}=-m(\vec{\omega}\times(\vec{\omega}\times\vec{r}))=-(m\omega^2 r)\hat{r}$

$\vec{F}_{cor}=-2m(\vec{\omega}\times\vec{v})$

Kepler's Second Law

$\frac{dA}{dt}=\frac{L}{2m}=\frac{v_o r_o}{2}$

Note that this implies that $\frac{L}{2m}T=\pi ab$

Since the area of an ellipse is πab

Kepler's Third Law

$T^2=\frac{4\pi^2a^3}{GM}$

Total Orbital Energy T+V

$E=-\frac{GMm}{2a}$

Escape Velocity

$v_{esc}=\sqrt{\frac{2GM}{R}}$

Orbital Velocity

$v_{orb}=\sqrt{\frac{GM}{R}}$

Euler-Lagrange Equations

L=T(x,v,t)-V(x)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}$

Differential Equations for Orbits

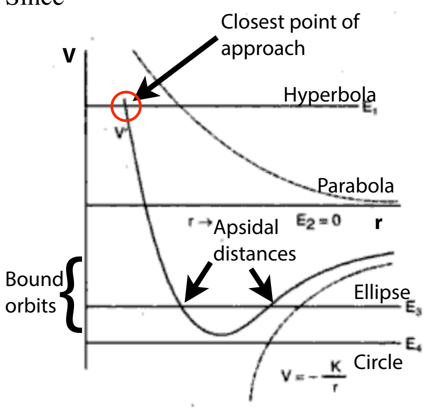
$m\ddot{r}+\frac{L^2}{mr^3}=-\frac{dV}{dr}=F(r)$

$E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)$

Equivalent Fictitious Potential

$V'=-\frac{k}{r}+\frac{L^2}{2mr^2}$

Since $-\int_r^\infty -\frac{L^2}{mr^3}=\frac{L^2}{2mr^2}$



E (straight lines) is the total energy of the orbiting particle, and cannot be less than V, because that would imply a negative KE.

Finding Central Force

1. Use Lagrangian to establish $m\ddot{r}-m\dot{\theta}^2r+\frac{dV}{dr}=0$
2. Note that $mr^2\omega$=L is a constant
3. Define U=1/r
4. $\frac{dU}{d\theta}=\frac{dU}{dr}\frac{dr}{dt}\frac{dt}{d\theta}=\frac{-m}{L}\dot{r}$
5. $\frac{d^2U}{d\theta^2}=\frac{d}{dt}\left(\frac{dU}{d\theta}\right)\frac{dt}{d\theta}=\frac{-1}{r^2}\frac{dr}{d\theta}$
6. Find the equation of motion r(θ)
7. Find dr/dθ, then $d^2U/d\theta^2$
8. Find d^2r/dt^2, sub into (1)
9. Solve for -dV/dr=F

Hamiltonian

p=Generalized momentum

q=Generalized coordinate

$\dot{p}=-\frac{\partial H}{\partial q}, \dot{q}=\frac{\partial H}{\partial p}$

H=T+V

$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}$

Gauss' Law for Gravity

$\oint \vec{g}\cdot d\vec{A}=-4\pi G m_{enclosed}$

Young's Modulus

E=tensile stress/tensile strain

$E=\frac{\left(\frac{F}{A}\right)}{\left(\frac{\Delta l}{l}\right)}$

$F=\frac{EA}{L}\Delta L=kx$

Shear Modulus

$G=\frac{\left(\frac{F}{A}\right)}{\left(\frac{\Delta x}{L}\right)}=\frac{\left(\frac{F}{A}\right)}{\tan\phi}$

Velocity of a shear wave (S-wave)

$v=\sqrt{\frac{G}{\varrho}}$

Velocity of a pressure wave (P-wave)

$v=\sqrt{\frac{K+\frac{4}{3}G}{\varrho}}$

Rigid Bodies

Torque

$\tau=r\times F$

Angular momentum

L=r x p=r x (mv)

L=Iw

Moment of Inertia

$I=\int_0^R r^2dm$

r: perpendicular distance from axis of rotation

Parallel Axis Theorem

$I=I_{CM}+MR^2$

Perpendicular Axis Theorem

$I_x+I_y=I_z$ for flat object in X-Y plane

Rolling without Slipping

$v=r\omega$

$a=r\alpha$

Relative velocity at contact point=0

Friction does no work

Gyroscopic motion

$dL=\tau dt$

Massless Cantilever

$F=-\frac{E\omega t^3}{4L^3}h=-m\omega^2h$

Coefficient of Restitution

$C_R=\frac{v_{1,f}-v_{2,f}}{v_{1,i}-v_{2,i}}$

$C_R=\sqrt{\frac{h}{H}}=-\frac{v_f}{v_i}$

Hydromechanics

Fluid Pressure

$P=P_0+\varrho gh$

Bernoulli's Equation

$P+\frac{1}{2}\varrho v^2+\varrho gy = constant$

Archimedes' Principle
$F_{buoyant}=\varrho gV$

Continuity Equation
$A_1v_1=A_2v_2$=Volume flow rate

Surface Tension
Sphere: $\Delta P=\frac{2\gamma}{R}$, γ=force per unit length

Stokes' Law
$F_{drag}=6\pi\mu rv$

r=Radius of falling spherical object
μ=Fluid medium viscosity (kg m^{-1} s^{-1})
v=Velocity of fall

Torricell's Theorem
Velocity of efflux of liquid=velocity if it were a particle falling from the surface of the liquid to the orifice.
$v=\sqrt{2gh}$

Capillary Rise
$h=\frac{2\sigma}{R\varrho g}\cos\theta$, R=radius of tube, σ=surface tension, θ=contact angle (20 deg for water on glass)

Critical Velocity for Laminar Flow
$V_c=\frac{k\eta}{\varrho r}$, k=Reynold's Number

Poiseuille's Equation
$\frac{dV}{dt}=\frac{\pi R^4}{8\eta}\frac{\Delta P}{\Delta L}$

Thermodynamics

Coefficient of Linear Expansion
$\alpha=\frac{1}{L_i}\frac{dL}{dT}$

$dL=\alpha L_i dT$

Coefficient of volume expansion
$\beta=\frac{1}{V_i}\frac{dV}{dT}$

$dV=\beta V_i dT$

$\beta=3\alpha$

Heat input
$dQ=mcdT$

Latent Heat
$L=\frac{Q}{M}$

Water (Fusion): 3.34×10^5 J/kg
Water (Vaporization): 2.26×10^6 J/kg
Ethanol (Fusion): 1.08×10^5 J/kg
Ethanol (Vaporization): 8.55×10^5 J/kg

First Law of Thermodynamics
$\delta Q=dU+\delta W$
$dQ=nC_v dT+pdV$
W=Work done by gas

Second Law of Thermodynamics
Clausius Statement: Heat cannot spontaneously flow from a material at a lower temp. to a higher temp.
Kelvin Statement: Impossible to convert heat completely to work in a cyclic process
$\Delta S_{universe}\geq 0$

Work
$W=\int_{V_i}^{V_f} PdV$

Heat Engine
Important parameter is work done
$e=\frac{W}{Q_H}=1-\frac{Q_c}{Q_H}=1-\frac{T_c}{T_H}(Carnot)$

Refrigerator
Important parameter is heat entry
$\eta=\left|\frac{Q_{in}}{W}\right|=\frac{Q_{in}}{Q_{out}-Q_{in}}=\frac{T_c}{T_H-T_c}$

Heat Pump
Important parameter is heat exit
$\eta=\frac{Q_{out}}{W}=\frac{Q_{out}}{Q_{out}-Q_{in}}=\frac{T_H}{T_H-T_c}$

General Coefficient of Performance
$\eta=\frac{\Delta Q}{W}$

ΔQ is the change in heat in the reservoir of interest
W is the work done in the process

Heat Conduction
$\frac{dQ}{dt}=-kA\frac{dT}{dx}$

Radiation Power
$P=\sigma eA\left(T^4-T_s^4\right)$

$\sigma=5.67\times10^{-8}$ $Wm^{-2}K^{-4}$

Adiabatic Condition
PV^{γ}=constant
$TV^{\gamma-1}$=constant
$P^{\gamma-1}T^{-\gamma}$=constant
Let $PV^{\gamma}=K$
Then $W=\int_{V_i}^{V_f} PdV=K\int_{V_i}^{V_f}\frac{dV}{V^{\gamma}}$

$W=\frac{K\left(V_f^{1-\gamma}-V_i^{1-\gamma}\right)}{1-\gamma}$

Ideal Gas Equations
$P=\frac{1}{3}nmv^2$

$E_{1\ atom}=\frac{3}{2}kT$

$E_{total}=\frac{3}{2}nRT=\frac{3}{2}pV$

Hydrostatic Equilibrium
No net force due to pressure differential
$dP=-\varrho gdz$

Star Luminosity
$L=4\pi d^2 f$

f=Measured flux at distance d
Power output in all directions of star

Particle in a box
$E_n=\frac{n^2h^2}{8mL^2}$

Evaluating the partition function q,
$q_L=\sum_{n=1}^{\infty} e^{-\beta\varepsilon_n}=\sum_{n=1}^{\infty} e^{-\beta(n^2-1)\varepsilon_1}=\int_0^{\infty} e^{-n^2\beta\varepsilon_1}dn$

$q_L=L\sqrt{\frac{2\pi m}{h^2\beta}}$ in 1 dimension
In 3 dimensions, $q=q_Xq_Yq_Z$

$q=\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}XYZ=\left(\frac{2\pi m}{h^2\beta}\right)^{\frac{3}{2}}V$

$q=\frac{V}{\Lambda^3}$, $\Lambda=h\sqrt{\frac{\beta}{2\pi m}}$

Lambda is the thermal wavelength: Roughly the average de Broglie wavelength of the particles in a gas

Boltzmann Distribution Law
$\frac{n_i}{N}=\frac{e^{-\beta\varepsilon_i}}{q}$, $q=\sum_j g_j e^{-\beta\varepsilon_j}$

$E(T)=-\frac{N}{q}\frac{dq}{d\beta}=-N\frac{d\ln q}{d\beta}$

$U(T)=U(0)+E(T)=U(0)-N\frac{d\ln q}{d\beta}$

Note that all derivatives are partials.
$\beta=\frac{1}{kT}$

Volume assumed to remain constant.
More complex:
$N(v)=4\pi N_o\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}v^2e^{\frac{-mv^2}{2kT}}$

Note that 1/2 mv^2=E

Molecular Speeds
Root Mean Square: $\sqrt{\frac{3kT}{m}}$

Mean: $\sqrt{\frac{8kT}{\pi m}}$

Maximum: $\sqrt{\frac{2kT}{m}}$

Otto Cycle
$e=1-\frac{1}{\left(\frac{V_1}{V_2}\right)^{\gamma-1}}$

Two isochores, two adiabats
V_1/V_2=Compression ratio > 1

Diesel Cycle
$e=1-\frac{1}{r^{\gamma-1}}\frac{(\alpha^{\gamma}-1)}{\alpha-1}\frac{1}{\gamma}$

r=adiabatic compression ratio
a=isobaric expansion ratio/cut-off ratio

Entropy
$S=k\ ln\ \omega$

W: number of microstates corresponding to one macrostate

$W=\frac{N!}{n_1!\,n_2!\,...\,n_x!}$

W is the weight of the configuration
For greatest W, $p_i = e^{-\beta\varepsilon_i}/q$

$\ln W = N \ln N - \sum_i N_i \ln N_i$

$S(T)=\frac{U(T)-U(0)}{T} + Nk \ln q$

When considering the gas as a whole, $Q=q^N$, E = energy of the whole system

$Q=\sum_i e^{-\beta E_i}$, $E_i=\varepsilon_1+\varepsilon_2...+\varepsilon_N$

$S(T)=\frac{U(T)-U(0)}{T} + k \ln Q$

$dS=\frac{dQ}{T}=\frac{dU}{T}+\frac{pdV}{T}=\frac{nC_v}{T}dt+\frac{nR}{V}dV$

$\Delta S = nC_v ln\left(\frac{T_2}{T_1}\right) + nRln\left(\frac{V_2}{V_1}\right)$

Adiabatic: dQ=0, dS=0

Isochoric: $\Delta S=nC_v ln\left(\frac{P_2}{P_1}\right)$

Isothermal: $\Delta S=nRln\left(\frac{V_2}{V_1}\right)$

Isobaric: $\Delta S=nC_p ln\left(\frac{V_2}{V_1}\right)=n(R+C_v)ln\left(\frac{V_2}{V_1}\right)$

Mean Free Path in a Gas
$L=\frac{1}{\pi n d^2\sqrt{2}}$

n=number of molecules per unit volume
d=diameter of the molecule

Oscillations and Waves

Condition for stable minimum in PE
$\frac{dF}{dx}<0\ or\ \frac{\partial^2 U}{\partial x^2}>0$

Condition for SHM
$\ddot{q}+\omega^2 q= constant$

Frequency of oscillation
$\omega=\frac{1}{M}\frac{\partial^2 U}{\partial x^2}\ at\ r=r_0$

$\omega=\sqrt{\frac{k}{m}}$

Beats
$f_{beat} = | f_1-f_2 |$

Standing Waves
Asin(kx-wt)+Asin(kx+wt)
Resultant=2Asin(kx)cos(wt)
Wave does not propagate in time

Natural frequencies
Closed pipe/Clamped on both ends

$f_n=\frac{n}{2L}v, n=1,2,3...$

Open on one end

$f_n=\frac{n}{4L}v, n=1,3,5...$

Normal Modes
Relative amplitudes given by the eigenvector of the k matrix
All particles have the same frequency
N particles have N normal modes

Eigenvalue λ
Satisfies det(A-λI)=0

Eigenvector
Satisfies Ax=λx

General Solution to x
$x=Asin(\omega t+\phi)$

Torsional Pendulum
$\tau=-k\theta=I\frac{\partial^2\theta}{\partial t^2}$

Damped oscillations
$\omega=\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2}, F_{damp}=-bv$

Q Factor
$Q=2\pi$ x (Energy stored in 1 cycle)/(Energy dissipated in 1 cycle)

$Q=\frac{\omega_o}{\Delta\omega}$

$\Delta\omega$=Full width at half mean of energy
For oscillator on a spring,

$Q=\frac{\omega_o}{\gamma}=\frac{\omega_o m}{b}=\frac{\sqrt{mk}}{b}$

Q is the ratio of amplitude of object to amplitude of driving force at resonance
$A_{max}=Q\eta$
RLC: Q=Ratio of reactance to resistance
$Q=tan(\phi)$

For Series RLC Circuit
(Recall that $\omega_o=\frac{1}{\sqrt{LC}}$ and $\gamma=\frac{R}{L}$)

$Q=\frac{1}{R}\sqrt{\frac{L}{C}}$

For Parallel RLC Circuit

$Q=R\sqrt{\frac{C}{L}}$

Solutions to driven oscillator
General differential equation
$\ddot{x}+\gamma\dot{x}+\omega_0^2 = F_{max}\cos(\omega t)$
has solutions

$x(t)=Ae^{-\frac{\gamma t}{2}}cos\left(t\sqrt{\omega_0^2-\frac{\gamma^2}{4}}+\phi\right)+x_{ss}(t), \omega_o>\frac{\gamma}{2}$

$x(t)=(A+Bt)e^{-\frac{\gamma t}{2}}+x_{ss}(t), \omega_o=\frac{\gamma}{2}$

$x(t)=Ae^{-\left(\frac{\gamma}{2}+\sqrt{\frac{\gamma^2}{4}-\omega_o^2}\right)t}+Be^{-\left(\frac{\gamma}{2}-\sqrt{\frac{\gamma^2}{4}-\omega_o^2}\right)t}+x_{ss}(t), \omega_o<\frac{\gamma}{2}$

Steady State solution
$x_{ss}=A(\omega)cos(\omega t-\delta(\omega))$

$A(\omega)=\frac{F_{max}}{\sqrt{\left(\omega_o^2-\omega^2\right)^2+\gamma^2\omega^2}}, tan\delta(\omega)=\frac{\gamma\omega}{\left(\omega_o^2-\omega^2\right)}$

Non-dispersive Wave Equation
$\frac{\partial^2}{\partial x^2}y(x,t)=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t)$

Solution to Wave Equation in Complex Form
$Y(x,t)=Re\left(Y_{max}e^{i(kx-wt)}\right)$

Velocities
String: $v=\sqrt{\frac{T}{\mu}}$

Gas: $v=\sqrt{\frac{K}{\varrho}}$

$K=-V\frac{\partial P}{\partial V}=\left|\frac{dP}{\left(\frac{dV}{V}\right)}\right|=\gamma P$

Air: 1.42×10^5 Pa
Water: 2.2×10^9 Pa

Wave Impedance
Ratio of

$Z=\left|\frac{T_y}{v_{transverse}}\right|=\left|\frac{T\frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}}\right|=\left|\frac{T}{v}\right|$

$Z=\sqrt{\mu T}=\mu v$

Units of impedance: kg/s

Joint strings of different impedance
$T_1=T_2, \frac{v_1}{v_2}=\frac{\left(\frac{T}{Z_1}\right)}{\left(\frac{T}{Z_2}\right)}=\frac{Z_2}{Z_1}$

Reflection Coefficient R
Such that resultant incident wave is
$f(x,t)=Acos(kx-wt)+RAcos(kx+wt)$
Where Acos(kx-wt)=incident wave and RAcos(kx+wt) is the reflected wave moving in the opposite direction

$R=\frac{A_{reflected}}{A_{incident}}=\frac{Z_1-Z_2}{Z_1+Z_2}$

Z_1 and Z_2 are the impedances of the "initial" string and "final" string
Negative value means that the pulse is inverted, i.e. when first string is lighter
Angular velocity is equal on both sides
Wave is continuous (no kinks) at the boundary

Transmission Coefficient T
Such that the transmitted wave is f(x,t)=TAcos(kx-wt)
Where the transmitted wave is the incident wave multiplied by T

$T=\frac{A_{transmitted}}{A_{incident}}=\frac{2Z_1}{Z_1+Z_2}$

Relationship between R and T
1+R=T

Energy in a mechanical wave

$K_\lambda = \frac{1}{4}\mu\omega^2 A^2 \lambda = U_\lambda$

$E = K_\lambda + U_\lambda = \frac{1}{2}\mu\omega^2 A^2 \lambda$

$P = \frac{E}{T} = \frac{1}{2}\mu\omega^2 A^2 v$

Dry Air Constants

M=0.0289645 $kgmol^{-1}$

Adiabatic constant=1.400 at 20C

Heat Capacities

$C_v = \frac{R}{\gamma - 1}, C_p = \frac{\gamma R}{\gamma - 1}$

$C_p = C_v + R$

Sound Waves

$\Delta P = -K\frac{\partial s}{\partial x}$

$\Delta P_{max} = \varrho v \omega s_{max}$

Parallels

A in strings = s in sound

y in strings = ΔP in sound

μ *in strings =* ϱ *in sound*

Intensity

$I = \frac{P}{A} = \frac{1}{2}\varrho v(\omega s_{max})^2 = \frac{(\Delta P_{max})^2}{2\varrho v}$

Thresholds

Hearing: 1.00 x 10^{-12} Wm^{-2}

Pain: 1.00 Wm^{-2}

Doppler Effect

$f_L = \left(\frac{v + v_L}{v + v_S}\right) f_s$

$f_L = \sqrt{\frac{1-\beta}{1+\beta}}\, f_s$

$f = \left(1 - \frac{v_{relative}}{c}\right) f_o$, $v_{relative} << c$

$\Delta f = \frac{-v_{relative}}{c} f_o = \frac{-v_{relative}}{\lambda_o}$

$f_o = \gamma f(1 + \beta\cos\theta)$ if direction of observation and velocity of observer are different. Derive from 4-vectors, photon ejected at angle from S'

Decibels

$\beta = 10\, log\left(\frac{I}{1.00x10^{-12}\, Wm^{-2}}\right)$

Velocity of Water Waves

Shallow (no dispersion):

$v_p = v_g = \sqrt{gh}$

Deep (dispersion relation):

$\omega^2 = gk(1 + k^2 A^2)$

Electrostatics

Coulomb's Law

$dF = \frac{1}{4\pi\varepsilon_o}\frac{Qdq}{r^2}[r]$

$dE = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r^2}[r]$

$dV = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r}$, scalar!

E= - ∇V

$V = -\int_{initial}^{final} E \cdot dl$

If E is constant, V= -Ed

Electric Dipole

p=qd, pointing from - to +

q is the charge separation

$V(r,\theta) = \frac{kpcos\theta}{r^2}$

$V(r,\theta) = \frac{1}{4\pi\varepsilon_o}\left(\frac{r \cdot p}{r^3}\right)$

$\vec{E}(r,\theta) = \frac{2kp\cos\theta}{r^3}\hat{r} + \frac{kp\sin\theta}{r^3}\hat{\theta}$

$\vec{E}(x,y) = \frac{3kp\sin\theta\cos\theta}{r^3}\hat{x} + \frac{kp(3\cos^2\theta - 1)}{r^3}\hat{y}$

$\vec{E} = \frac{1}{4\pi\varepsilon_o r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$

Torques and Forces on a dipole

$\tau = p \times E$

$U = -p \cdot E$

$F = -\nabla U = \nabla(p \cdot \mathrm{E})$

Electric Flux

φ_E=E•A=∫E•dA

Gauss' Law

∮E•dA=q_{emc}/ε_o

$\nabla^2 V = -\varrho/\varepsilon_o$

$V = \int_{volume} \frac{k\varrho}{r} dv$

$E = \int_{volume} \frac{k\varrho}{r^2} dv\, [r]$

In terms of displacement vector D:

$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

$\vec{\nabla}\cdot\vec{D} = \rho_v$

$\oiint \vec{D}\cdot d\vec{A} = Q$

Solid Angle

$\Delta\Omega = \frac{A_1}{r_1^2}$, where A_1 is the area of the segment of the sphere subtending the angle

Total solid angle in a sphere: 4π

$\Delta\Omega = \frac{A_1 \cdot [r]}{r_1^2} = \frac{A_1\cos\theta}{r_1^2}$, if inclined

More specifically,

$\Omega = \int_S \frac{r \cdot dA}{|r|^3}$

Normal Electric Field on conductor surface

$E_{normal} = \sigma/\varepsilon_o$ because no electric field inside the conductor

Dielectrics

P is for Polarization: Dipole moment per unit volume

$P = \varepsilon_o \chi_e E$

Bound charge surface density σ_b

$\sigma_b = \vec{P}\cdot\hat{n}$

Volume charge density ϱ_b

$\varrho_b = -\nabla \cdot P$ (non uniform polarization)

Electric Field in a Dielectric

$E = E_{in} + E_{outside}$

$E_{in} = -\frac{1}{3\varepsilon_o}P$

Electric Displacement

$D = \varepsilon_o E + P$

Gauss' Law with Dielectrics

$\nabla \cdot D = \varrho_f$

$\oint D \cdot dA = Q_{free,enclosed}$

ϱ_f =Free charge density

Q_{enc}=Total free charge enclosed

Dielectrics Boundary Conditions

$D_{n,1} = D_{n,2}$

Displacement vector in the normal direction is the same

$E_{t,1} = E_{t,2}$

E-field in the tangential direction is the same

$H_{t,1} = H_{t,2}$

Reflection Coefficient

$\Gamma = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$

Note these are impedances.

Transmission Coefficient

$\tau = \frac{2\eta_2}{\eta_1 + \eta_2}$

Note these are impedances.

Energy of a charged configuration

$W = \frac{1}{2}\int_{volume} \varrho V d(volume)$ or line or area

$W = \frac{1}{2}\varepsilon_o \int_{all\ space} E^2 d(volume)$ and therefore

Energy Density: $\frac{dW}{d(volume)} = \frac{1}{2}\varepsilon_o E^2$

$W = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i)$, $V(r_i)$ is the potential due to all the other charges

Conductors

There are no E-fields in a conductor

All charge resides on the surface

Capacitors

$C=Q/V=dQ/dV$

Atomic Polarizability
$p=\alpha E$, E=applied E field

Dielectrics
$\varepsilon=\varepsilon_o(1+X_e)$, X_e=electric susceptibility
$\varepsilon_r=\varepsilon/\varepsilon_o=1+X_e$
ε_r=dielectric constant

Laplace's Equation
In free space, potential
$\nabla^2 V=0$
Properties:
1. Average of adjacent values
2. No local maxima or minima
3. Unique based on boundary

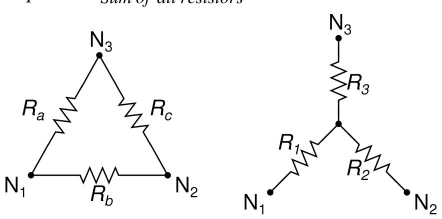
Electrodynamics
Ohm's Law
$\vec{J} = \sigma\vec{E}$

Discrete Current
$I=nqvA$
v=drift velocity
n=charge carrier density
J=nqv, vector
$I = \iint \vec{J} \cdot d\vec{A}$
$\nabla \cdot J= -\frac{\partial}{\partial t}\varrho_v$
$\oiint J \cdot dA = -\frac{\partial}{\partial t}Q_{enc}$

Drift Velocity
$v_d=\frac{qE}{m}\tau$
so $J=\sigma E=E/\rho$, or
$\varrho=\frac{m}{nq^2\tau}$

Current Density
$J=\frac{I}{A}=nqv=\frac{d\sigma}{dt}$
σ=Surface charge density

Delta to Y Transformation
$R_Y=\frac{(\text{Product of adjacent resistors})}{\text{Sum of all resistors}}$



$R_1=\frac{R_A R_B}{R_A+R_B+R_C}$
$R_2=\frac{R_B R_C}{R_A+R_B+R_C}$
$R_3=\frac{R_A R_C}{R_A+R_B+R_C}$

Y to Delta Transformation
$R=\frac{\text{Sum of all products}}{R_{opposite}}$
$R_A=\frac{R_1R_2+R_2R_3+R_1R_3}{R_2}$

Temperature coefficient of resistivity
$\alpha=\frac{1}{r_o}\frac{d\varrho}{dT}\sim\frac{1}{r_o}\left(\frac{\varrho-\varrho_o}{T-T_o}\right)$

Capacitor Charging
$I(t)=\frac{\varepsilon}{R}e^{-\frac{t}{RC}}=\frac{\varepsilon}{R}e^{-\frac{t}{\tau}},\tau=RC$
Discharging a capacitor
$I(t)= -\frac{Q}{RC}e^{-\frac{t}{RC}}= -\frac{Q}{\tau}e^{-\frac{t}{\tau}}$, $\tau=RC$
$I(t)= -\frac{Q}{RC}e^{-\frac{t}{RC}}= -\frac{V_o}{R}e^{-\frac{t}{\tau}}$, $\tau=RC$

Shockley diode equation
$I=I_s\left(e^{\frac{qV}{nkT}}-1\right), V_T=\frac{kT}{q}$
I_s=Reverse bias saturation current
V=Applied voltage
n=Non-ideality factor/emission coefficient, $\in(0,1]$
V_T=Thermal Voltage

Diode dynamic resistance (non-ohmic)
$R=\frac{dV}{dI}$

Superposition Theorem
The net current is the sum of the individual currents when each current source is considered separately (all others are open circuits)
The net voltage is the sum of all the individual voltages when each voltage source is considered separately (all others are shorted out)

Continuous resistance system
To find resistance or potential difference
1. Find J, current density
2. $J=\sigma E$
3. $V= -\int E \cdot dr$
4. V=RI

Capacitors
Q=CV
$U=Q^2/2C=Q/2V=1/2\ CV^2$
Series Capacitors: $\frac{1}{C_{eq}}=\sum_{i=1}^{N}\frac{1}{C_i}$
Parallel Capacitors: $C_{eq}=\sum_{i=1}^{N}C_i$

Kirchoff's Rules
$\sum I=0$ (at any junction)
$\sum V=0$ (any closed loop)
Constant magnetic flux only!
When moving in opposite direction as current in resistor, positive
Follow uphill/downhill analogy for long and short line of battery

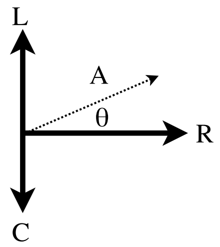
Rectified Average Current
Total charge that flows during a whole number of cycles is the same as though the current was constant at I_{rav}
$I_{rav}=\frac{2}{\pi}I=0.637I$

Root Mean Square values
$RMS[A\sin(wt)]=A/\sqrt{2}$
$RMS[A\cos(wt)]=A/\sqrt{2}$
RMS[Square wave]=A
$RMS[\text{Sawtooth}]=A/\sqrt{3}$

Reactance
Capacitative: $X_C= -i/(wC)$
Inductive: $X_L= iwL$

Phase Leading
Resistor: In phase
Inductor: Leads I by 90°
Capacitor: Lags I by 90°
If $i=I_o\sin(\omega t)$
$v=V_o\sin(\omega t+\varphi)$.
When φ is positive, the graph is translated to the left. Hence its starting point comes earlier, and v leads i if φ is positive.

Phasor Diagram



Phasor Operations
Let $Z_1= r_1\angle\theta_1$ and $Z_2= r_2\angle\theta_2$
$Z_1Z_2=r_1r_2\angle(\theta_1+\theta_2)$
$Z_1/Z_2=(r_1/r_2)\angle(\theta_1-\theta_2)$
$1/Z=(1/r)\angle(-\theta)$
$Z^*= r\angle(-\theta)$
$Z_1= r_1\angle\theta_1=r_1\cos(\omega t+\theta_1)$

Impedance
$Z=\sqrt{R^2+(X_L-X_C)^2}$
Amplitude of voltage: $V_{max}=I_{max}Z$
$V_{rms}=I_{rms}Z$

Phase Angle
$\tan(\phi)=\frac{X_L-X_C}{R}$
Net voltage $v=V_{max}\cos(wt+\varphi)$

Complex Form

Impedance can be written as R+iX
Re(Z)=Resistance R
Im(Z)=Reactance X
Inductive reactance is +iX
Capacitative reactance is -iX
E.g. Z=60+30i has resistance 60Ω and 30Ω reactance that is more inductive
Phase angle = arg(Z)
Magnitude= $\sqrt{(R^2+X^2)}$

AC Power
$P_{ave}=1/2\ V_{max}I_{max}\cos\varphi$
$=V_{rms}I_{rms}\cos\varphi$
When φ=0, purely resistive
cos φ=Power factor
Note that a pure inductor/capacitor develops no power
Note that w is angular, w=2πf

Lissajous Diagram
Parametric plot with:
x=Asin(wt)
y=Asin(wt-φ)

Angle	Shape	Direction
0	Line	1&3 quad.
90	Circle	ACW
180	Line	2&4 quad.
270	Circle	CW

Parallel AC Circuits
$\frac{1}{Z}=\frac{1}{Z_1}+\frac{1}{Z_2}$

Magnetic Field
Lorentz Force
F=q[E+vxB]

Cyclotron
$mv=qBr$ even in relativistic case
$r=\frac{mv}{qB}$
$\omega=\frac{qB}{m}$ =cyclotron frequency

Force on current carrying conductor
dF=I dl x B
$F=\int Idl\times B$

The magnetic force on any current carrying path is equal to the force exerted on an imaginary straight wire connecting the endpoints. (Net force=0 in a uniform field)

Biot-Savart Law
$$\vec{B}=\frac{\mu_o}{4\pi}I\int\frac{d\vec{l}\times\hat{r}}{r^2}$$
$$\vec{B}=\frac{\mu_o}{4\pi}\int\frac{dq\vec{v}\times\hat{r}}{r^2}$$
Single moving charge:
$$\vec{B}=\frac{\mu_o}{4\pi}\frac{q}{r^2}v\times\hat{r}$$
$$\vec{H}=\frac{1}{4\pi}\int\frac{d\vec{l}\times\hat{r}}{r^2}$$

Ampere's Law
∇xB=μ_oJ, no change in E flux
$\oint B\bullet dl=\mu_o I_{enc}$
$\oint\vec{H}\cdot\vec{dl}=I$

Displacement Current
$q=CV=\frac{\varepsilon A}{d}(Ed)=\varepsilon EA=\varepsilon\Phi_E$
$i_D=\varepsilon\frac{d\phi_E}{dt}$
$i_D=i_C$ for a charging capacitor

Generalized Ampere's Law
$\oint Bdl=\mu_o(i_c+i_D)_{enc}$

Magnetic Fields of:
Infinite wire: $B=\frac{\mu_o I}{2\pi r}$
Wire loop: $B=\frac{\mu_o I}{2r}$
Wire loop segment: $B=\frac{\mu_o I}{4\pi r}\theta$
Toroid: $B=\frac{\mu_o NI}{2\pi r}$
Solenoid: $B=\mu_o nI=\mu_o\left(\frac{N}{L}\right)I$

Magnetic Dipole
F=∇(μ•B)
τ=μxB
Magnetic moment $\mu=I\oint dA=IA$
Multiply by N for N turns
U= -μ•B
Direction of magnetic dipole: Same direction as wire's B-field (use right hand rule)

Atomic Magnetic Moments
Orbital magnetic moment of an electron is proportional to its orbital angular momentum
$\mu=\left(\frac{q}{2m}\right)L$
$\frac{\mu}{L}=\gamma$, gyromagnetic ratio
Spin magnetic moment of an electron is an integer multiple of the Bohr magneton
$\mu_B=\frac{q\hbar}{2m_e}$, i.e. when L=$\hbar$

H Field
$B=\mu_o H+\mu_o M$
M=magnetization, magnetic dipole moment per unit volume
$B=\mu_o(H+M)$
$B=\mu H$
$\mu=\mu_r\mu_o$
$\mu_r=1+X_m$
$M=X_m H$

MAXWELL'S EQUATIONS
$\nabla\times E=-\frac{\partial B}{\partial t}$
$\nabla\times H=J+\frac{\partial D}{\partial t}$
$\nabla\cdot D=\varrho_v$
$\nabla\cdot B=0$

Type of magnetism
$X_M<0$: Diamagnetic
$X_M>0$, small: Paramagnetic
X_M large: Ferri/ferromagnetic

Hysteresis Loops
Plot Magnetization M/Gauss against external H field/Oe
Intercepts on M axis: Remanence B_r
Intercepts on H axis: Coercive Force H_c
Area within loop: magnetic energy loss per unit volume per cycle
Energy Product: Area of largest B-H rectangle within 2nd quadrant (top left hand corner) of the curve
Units: kJ/m^3
1 MGOe=7.96 kJ/m^3
Oersted: Ampere-turns/m or A/m
Unit of H field

Magnetization dependence on temperature for paramagnetic material
$M=C\frac{B}{T}$
C=Curie constant

Hall Voltage
A current-carrying conductor moving through a magnetic field will experience a Hall Voltage between the top and bottom of the conductor
$\Delta V_H=\frac{IBd}{nqA}=\frac{IB}{nqt}=\frac{IB}{t}\left(\frac{1}{nq}\right)=\frac{IB}{t}R_H$
d=height of conductor
t=thickness of conductor in direction of magnetic field
R_H=Hall coefficient=1/nq
Similar to velocity selectors
$v_d=\frac{E_H}{B}$

Magnetic Flux
$\Phi_B=\int B.dA$

Faraday's Law
$\oint E\bullet dl=-d\varphi_B/dt$
Through the surface defining the path

Motional EMF
ε= - vBL
General: ε=∮(v x B) •dl for closed conducting loop

Lenz's Law
Induced current and induced emf in a conductor are in a direction to set up a B field that would oppose the change that produced them

RL circuits
To use the fake Kirchoff's Law at an instant in time, voltage drop across an inductor is Ldi/dt, direction similar to resistor
With emf source:
$I(t)=\frac{\varepsilon}{R}\left(1-e^{-\frac{Rt}{L}}\right)=\frac{\varepsilon}{R}\left(1-e^{-\frac{t}{\tau}}\right), \tau=\frac{L}{R}$

LC Circuits
Oscillation with angular frequency $\omega=\frac{1}{\sqrt{LC}}$ at resonance, i.e. $X_C=X_L$

Inductors in Series and Parallel
Just like resistors.
Use "potential drop" to prove.

RLC Circuits
$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=0$
$\omega=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$, underdamped

Inductor Energy
$\frac{dU}{dt}=\left(L\frac{dI}{dt}\right)I$=*Rate energy is stored*
$U=\frac{1}{2}LI^2$

Magnetic Energy Density
$\frac{dU}{dV}=\frac{B^2}{2\mu_o}$ in vacuum
In a medium: $\frac{dU}{dV}=\frac{B^2}{2\mu}$

Self Inductance
$L=\frac{N\phi_B}{I}$
Total magnetic flux linkage induced by a certain current, divided by that current

Mutual Inductance
$\sqrt{L_1L_2}=M$
$M=\frac{N_1\phi_1}{I_2}=\frac{N_2\phi_2}{I_1}$
$\varepsilon_1=-M\frac{di_2}{dt}, \varepsilon_2=-M\frac{di_1}{dt}$

Back EMF
$\varepsilon=-L\frac{di}{dt}$

Transformers
$\frac{V_2}{V_1}=\frac{N_2}{N_1}$
$V_1I_1=V_2I_2$=*Power delivered*

EM Waves
Relating E Field and H Field
$$\vec{H}=\hat{a}_p\times\frac{\vec{E}}{\eta}$$

Wave impedance
$\eta=\sqrt{\frac{\mu}{\varepsilon}}=\mu_o c\sqrt{\frac{\mu_r}{\varepsilon_r}}$

Wave Properties in a Lossless medium
Condition for lossless: σ=0
$\eta=\frac{|E|}{|H|}$
$\lambda=\frac{2\pi}{k}$
$v_p=\frac{\omega}{k}$
$n=\frac{c}{v_p}$
$k=\frac{\omega}{v_p}=\omega\sqrt{\mu\varepsilon}$

Lossy Medium
σ≠0
In phasor form,
$\nabla\times H=j\omega\varepsilon_c E$
$\varepsilon_c=\varepsilon-\frac{\sigma j}{\omega}$
ε_c is the complex permittivity
$\eta_c=\sqrt{\frac{\mu}{\varepsilon-\frac{j\sigma}{\omega}}}$
η_c is the complex impedance

Loss Tangent
$tan\delta=\frac{\left(\frac{\sigma}{\omega}\right)}{\varepsilon}$
Ratio of the imaginary part of the complex permittivity and the real part

Complex Propagation Constant
$\gamma=\sqrt{j\omega\mu(\sigma+j\omega\varepsilon)}$
In rectangular form,
$\gamma=\alpha+\beta i$
α is the attenuation constant
Units of α=NP/m=Neper per meter
β is the phase constant
Units of β=Rad/m

Losses in a conductive medium
Condition for conductor: $\sigma\geq 20\omega\varepsilon$
To first order,
$\gamma=\sqrt{\frac{\omega\mu\sigma}{2}}+j\sqrt{\frac{\omega\mu\sigma}{2}}$
$\eta_c=e^{j\frac{\pi}{4}}\sqrt{\frac{\mu\omega}{\sigma}}$
$v_p=\frac{\omega}{\beta}$
$\lambda=\frac{2\pi}{\beta}$
General Equation (one dimension)
$\vec{E}=|E_o|e^{-\alpha z}\cos(\omega t-\beta z+\phi)\hat{i}$

Skin Depth
$\delta=\frac{1}{\alpha}=\sqrt{\frac{2}{\omega\mu\sigma}}$
Represents the average depth at which AC current flows. AC current tends to distribute itself on the surface, hence increasing resistance.
Resistance of a thick slab (thickness >> δ) to AC is exactly equal to the resistance of a slab with thickness δ to DC current.

Types of Polarization
Given an electric field of general form:
$\vec{E}=|E_x|\cos(\omega t-kz+\phi_x)\hat{i}$
$+|E_y|\cos(\omega t-kz+\phi_y)\hat{j}$
Evaluating at z=0,
$\vec{E}=|E_x|\cos(\omega t+\phi_x)\hat{i}+|E_y|\cos(\omega t+\phi_y)\hat{j}$
When $\varphi_x-\varphi_y=n\pi$, linear polarization
When $|E_x|=|E_y|$ and $\varphi_x-\varphi_y=n\pi+\pi/2$, circular polarization
If $\varphi_x-\varphi_y=\pi/2$, right handed circular polarization
If $\varphi_x-\varphi_y=-\pi/2$, left handed circular polarization
All other cases: Elliptical polarization

Relation between E and B
E=cB
In medium, E=vB

Velocity
$v=\frac{1}{\sqrt{\varepsilon\mu}}=\frac{1}{\sqrt{\varepsilon_r\varepsilon_o\mu_r\mu_o}}=\frac{c}{\sqrt{\varepsilon_r\mu_r}}=\frac{c}{n}$
$n=1/\sqrt{(\varepsilon_r\mu_r)}$

Energy Density
$\frac{dU}{dV}=\frac{1}{2}\varepsilon_o E^2+\frac{1}{2\mu_o}B^2=\varepsilon_o E^2=\frac{B^2}{\mu_o}$
E and B are average values
E field and B field contribute equally to a wave's energy

Poynting Vector
Magnitude and direction of energy flow rate, i.e. Intensity. Units: W/m^2
$$\vec{S}=\vec{E}\times\vec{H}=\frac{1}{\mu_o}\vec{E}\times\vec{B}$$
$|S_{avg}|=E_{max}B_{max}/2\mu_o=cu_{avg}$
u_{avg}=Average energy density
Power=∮S•dA

Time Averages

Given $A(t)=A_o\cos(\omega t+\varphi_1)$ and $B(t)=B_o\cos(\omega t+\varphi_2)$

$<A(t)B(t)> = \frac{A_oB_o}{2}\cos(\phi_1-\phi_2)$

In phasor form,

$$< A(t)B(t) >= \frac{\mathrm{Re}(\vec{A}\cdot\vec{B}^*)}{2}$$

Angular Momentum of a Photon in a circularly polarized beam

$|\frac{L}{E}|=\frac{1}{\omega}$, E=energy

Oscillating electric dipole

For $q=q_o\cos(wt)$ and $p=p_o\cos(wt)$

Power $P=\frac{\mu_0 p_0^2 w^4}{12\pi c}$

P_0=Maximum dipole moment

Potential

$$V(r,\theta,t)=-\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Approximations for perfect electric dipole

1. d << r
2. d << c/w
3. r >> c/w

Since $\lambda=2\pi c/w$, $d<<\lambda$

Larmor Radiation Formula

Power radiated by an accelerating charge

$$P=\frac{\mu_o q^2a^2}{6\pi c}=\frac{q^2a^2}{6\pi\varepsilon_o c^3}=\frac{2}{3}\frac{kq^2a^2}{c^3}$$

Relativistic:

$$P=\frac{\mu_o q^2a^2\gamma^6}{6\pi c}=\gamma^6 P_{non\text{-}relativistic}$$

Coupling Parameter for Plasmas

$$\Gamma=\frac{n^{\frac{1}{3}}e^2}{4\pi\varepsilon_o kT}$$

Mean interparticle distance x Typical potential energy of interaction / kT

Radiation Pressure

Absorbing: P=I/c

Reflecting: P=2I/c

Twin Slit

Bright: $d\sin\theta=m\lambda$

Dark: $d\sin\theta=\left(m+\frac{1}{2}\right)\lambda$

$y=R\frac{m\lambda}{d}$ for small angles

Path Difference

$\frac{\delta}{\lambda}=\frac{\phi}{2\pi}$

$\delta=d\sin\theta$

$E_{net}=E_o\sin(\omega t)+E_o\sin(\omega t+\phi)$

$E_{net}=E_o+E_oe^{-i\phi}$

$I(\theta)\propto E(\theta)E^*(\theta)$

$I=I_{max}\cos^2(\varphi/2)$

$=I_{max}\cos^2(\pi d\sin\theta/\lambda)$ far, far away

Intensity of twin slit including single slit diffraction

$$I=I_{max}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\left[\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\frac{\pi a\sin\theta}{\lambda}}\right]^2$$

d=distance between slits

a=width of a single slit

Phase Changes

Reflection from higher n: 180°

Reflection from lower n: 0°

Thin Film Interference

Constructive: 2nt=path difference

Single Slit

$$I=I_{max}\left[\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\frac{\pi a\sin\theta}{\lambda}}\right]^2$$

Minima: $a\sin\theta=m\lambda$

Rayleigh Criterion

Expression for smallest angular separation for resolution

Slit: $\theta_{min}=\frac{\lambda}{a}$

Circular aperture: $\theta_{min}=\frac{1.22\lambda}{D}$

Note that D is the aperture diameter

X-ray diffraction (Bragg)

Constructive interference

$2d\sin\theta=m\lambda$, m=1,2,3…

d=lattice spacing

θ is measured from the horizontal

Optics

Snell's Law

$n_1\sin\theta_1=n_2\sin\theta_2=n_i\sin\theta_i$ along ray

Refractive index

$n_i=c/v_i$

Thin Lens formula (Gaussian)

1/s + 1/s' = 1/f = P

Positive: Real

Negative: Virtual

Relating small changes in s and s'

Implicit differentiation with respect to s

$$\frac{ds'}{ds}=-\left(\frac{s'}{s}\right)^2$$

Thin Lens formula (Newtonian)

$xx' = f^2$

x=distance of object from focal point

x'=distance of image from other focal point

Object: Left of focal point is positive

Image: Right of focal point is positive

General form, medium on both sides different (Newtonian)

xx'=ff'

Magnification

m = y'/y = -s'/s

Positive: Upright

Negative: Inverted

Angular Magnification

$m=\theta/\theta_o$

Lens makers' formula

$$P=\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

Convex: Positive R

Concave: Negative R

Power of individual surfaces

$P=(n_{right}-n_{left})/R$

Power of lens=Sum of powers of individual surfaces

Thin lens system

Power of system is equal to sum of individual powers of lens components

$P_{net}=P_1+P_2$

Thin lens system with separation

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}$$

Refraction at spherical interface

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{R}$$

$$P=\frac{n_2-n_1}{R}$$

s and s' measured from first incident surface

Thick lens formula

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n-1)d}{nR_1R_2}\right)$$

d=thickness of lens

Compound microscope

$$M=-\frac{L}{f_o}\left(\frac{25\,cm}{f_e}\right)$$

f_o=focal length of objective

f_e=focal length of eyepiece

L=distance between lenses

Brewster's Law

Reflected light is completely polarized when the angle between reflected and refracted beams is 90°

$\tan\theta_b=n_2/n_1$

Types of polarization
P-like (transverse magnetic TM)
Parallel to the plane
"Plunge through"
S-like (transverse electric TE)
Perpendicular to the plane
"Skip" off

Fresnel's Equations

$$R_s=\left(\frac{n_1\cos\theta_i-n_2\cos\theta_t}{n_1\cos\theta_i+n_2\cos\theta_t}\right)^2$$

$$R_p=\left(\frac{n_1\cos\theta_t-n_2\cos\theta_i}{n_1\cos\theta_t+n_2\cos\theta_i}\right)^2$$

Angle of Minimum Deviation (Prism)
Entering angle and exiting angle are the same
Ray travels perpendicular to the bisector of the apex angle

$$n=\frac{\sin\left(\frac{D+\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Quantum

Compton Shift

$$\Delta\lambda=\frac{h}{m_e c}(1-\cos\theta)$$

De-Broglie Wavelength

$$\lambda=\frac{h}{p}$$

Planck's Distribution Law

$$u(f,T)=\frac{8\pi hf^3}{c^3}\frac{1}{e^{\frac{hf}{kT}}-1}$$

$$u(\lambda,T)=\frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$

$$I(f,T)=\frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{kT}}-1}$$

I is Power per unit area of emitting surface per unit solid angle, per unit frequency.
Integrate across all frequencies to find net power output per unit area of surface per unit solid angle.

Scattering
For stationary central force:
$b=\frac{zz'e^2}{2E}\cot\frac{\theta}{2}$, z and z' are relative charges of incident particle and central force, E is total energy, θ is scattering angle

Nuclear Spin
Even atomic number: Integer Spin
Odd atomic number: Half-integer spin
Even proton number, even neutron number: Zero spin

Magnetic Dipole Moment

$$\mu=g\frac{e}{2mc}S$$

g=Landé Factor (about 2 for e^-)
S=Intrinsic spin ($1/2\hbar$ for e^-)

Random Walk
Given $\vec{D}=\vec{l}_1+\vec{l}_2+\vec{l}_3...+\vec{l}_N$
$\sqrt{<D^2>}=l\sqrt{N}$, l=distance moved per step, N=number of steps

Relativity

Lorentz Transformation

$$x'=\gamma(x-vt)$$
$$y'=y$$
$$z'=z$$
$$t'=\gamma\left(t-\frac{vx}{c^2}\right)$$

4 vector Lorentz transform
Primes indicate moving frame with relative velocity β
$A_1'=\gamma(A_1+i\beta A_4)$
$A_2'=A_2$
$A_3'=A_3$
$A_4'=\gamma(A_4-i\beta A_1)$

$A_1=\gamma(A_1'-i\beta A_4')$
$A_2=A_2'$
$A_3=A_3'$
$A_4=\gamma(A_4'+i\beta A_1')$

Rapidity
Rapidities add.
$\theta=\tanh^{-1}(\beta)$
$\theta_{net}=\theta_1+\theta_{2+}...$
$\beta_{net}=\tanh(\theta_{net})$
$E=mc^2\cosh\theta$
$|P|=mc\sinh\theta$

Proper Time
$t=\gamma\tau$
Proper time τ is the shortest possible interval of time, compared to other frames of reference.

Length Contraction

$$l=\frac{l_o}{\gamma}$$

4 velocity
$u=\gamma(u_3,ic)$

4 wave vector
K=(K,iK)

Four Momentum

$$P_\neg=\left(P,\frac{iE}{c}\right)$$

$P_\neg\cdot P_\neg=-m_0^2c^2$ note negative
For massless particles:
P=(E/c, iE/c), P•P=0
Linear momentum: $P=\gamma m_o v$

Minkowski Diagrams
Plot ct against x
Gradient is c/v=1/β
Moving frames can be represented by shifted axes making an angle of $\tan^{-1}\beta$ with the original axes
Photons make an angle of 45°
Particles at rest are straight lines

Lorentz Transformation for Fields
$E_x'=E_x$
$E_y'=\gamma(E_y-vB_z)$
$E_z'=\gamma(E_z+vB_y)$
$B_x'=B_x$
$B_y'=\gamma\left(B_y+\frac{v}{c^2}E_z\right)$
$B_z'=\gamma\left(B_z-\frac{v}{c^2}E_y\right)$

Matter

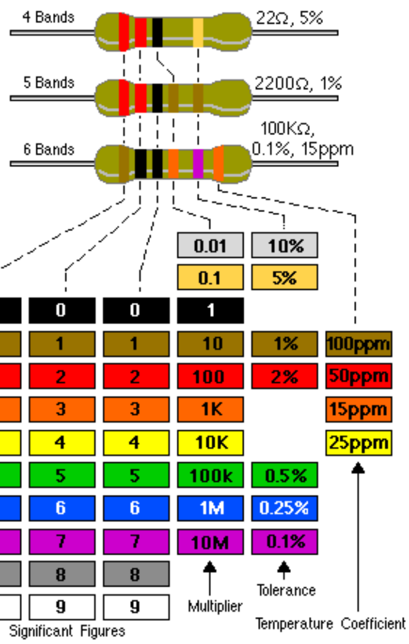
Alpha Decay
Nucleus releases He^{2+}, reduce proton and ne utron number by 2 each

Beta Plus Decay
Energy + p → n + e^+ + v_e
v_e=electron neutrino
Since neutron mass is greater than proton mass, the mother nucleus has to have a smaller binding energy than the daughter nucleus
Proton number decrease by one (plus=too many protons)

Beta Minus Decay
n → p + e^- + $\tilde{v}_e$
$\tilde{v}_e$ = electron antineutrino
Proton number increase by one (minus=too little protons)

Mass Defect
Mass defect = Unbound total mass - Measured bound mass
Binding energy = Mass defect x c^2
Nickel-62 has the highest binding energy, then Iron-58 and Iron-56

Hydrogen Transitions
Lyman: n=1
Balmer: n=2
Paschen: n=3
Brackett: n=4
Pfund: n=5
α: 1 level difference
β: 2 level difference
γ: 3 level difference etc.



Resistor Color Code System